Queen Mary
University of London

M.Sc. EXAMINATION

ASTMO41 Relativistic Astrophysics

Monday, 9 May 2005
10:00-11:30

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators ARE permitted in this examination. The unauthorized use of material stored in a pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

You are reminded of the following.

Physical Constants

Gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
Speed of light	c	3×10^8 m s^{-1}
Solar mass	$M_\odot$	2×10^{30} kg
Solar radius	$R_\odot$	7×10^5 km
1 kpc		3.09×10^{19} m

Notation

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ and take on the values 1, 2, 3.

Four-dimensional tensor indices are denoted by Latin letters $i, j, k, l, ...$ and take on the values 0, 1, 2, 3.

The metric signature (+ - - -) is used.

Useful formulae

The following results may be quoted without proof:

Hamilton-Jacobi equation:

$$g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} - m^2c^2 = 0,$$

where four-momentum $p_i = -\frac{\partial S}{\partial x^i}$ and $p_0 = E$ (energy), $p_3 = L$ (angular momentum).

Schwarzschild metric:

$$ds^2 = \left(1 - \frac{r_g}{r}\right)c^2dt^2 - \frac{dr^2}{\left(1-\frac{r_g}{r}\right)} - r^2\left(\sin^2\theta d\phi^2 + d\theta^2\right).$$

Gravitational radius of a body of the mass M: $r_g = 2GM/c^2 = 3(M/M_\odot)$ km.

Kerr metric:

$$ds^2 = (1 - \frac{r_g r}{\rho^2})c^2dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2 +$$

$$\frac{2r_g rac}{\rho^2}\sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, $a = \frac{J}{Mc}$ and J is angular momentum.

For the Schwarzschild and Kerr metric: $x^0 = ct$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$.

Quadrupole formula for the metric perturbations associated with gravitational waves:

$$h_{\alpha\beta} = -\frac{2G}{3c^4R}\frac{d^2D_{\alpha\beta}}{dt^2},$$

where R is the distance to the source of the gravitational waves and

$$D_{\alpha\beta} = \int(3x_\alpha x_\beta - r^2\delta_{\alpha\beta})dM$$

is the quadrupole tensor of the source.

Gravitational wave luminosity of circular binary system:

$$L_{gw} = -\frac{dE}{dt} = \frac{32Gm^2\omega_0^6r^4}{5\pi c^5}.$$

SECTION A

Each question carries 20 marks. You should attempt ALL questions.

1. (a) A spacecraft exploring a planet of mass m and radius r moves around the planet along a circular orbit of radius $R = 6r$. Ignoring the transverse Doppler effect, evaluate the redshift z of the radio signal emitted by a probe left on the surface of the planet and received by the spacecraft.

(b) Another spacecraft moves very far from any gravitating bodies with acceleration a. The redshift of a photon emitted at the bottom of the rocket and detected at its top is $z^{'} \approx 10^{-14}$. Evaluate the acceleration a if the height of the rocket is 100 m. [Hint: First calculate the gravitational redshift when $R - r \ll r$ and then apply the equivalence principle.]

2. (a) A star forms a black hole of mass M. Show that at the moment when the radius of the star is equal to $10^3 r_g$ its average density is

$$\rho \approx 2 \times 10^{10} \text{ kg m}^{-3} \left(\frac{M}{M_\odot}\right)^{-2} .$$

(b) Using simple Newtonian estimates, show that to an order of magnitude the radius of tidal disruption for a star of mass m and radius r in the gravitational field of a black hole of mass M is

$$R_{TD} \approx r \left(\frac{M}{m}\right)^{1/3} .$$

Compare this radius with the gravitational radius and find the black hole mass for which $R_{TD} = 10^3 r_g$. Give the answer in solar masses.

3. (a) An observer moves along a circular orbit of radius r in the equatorial plane ($\theta = \pi/2$) of a rotating black hole. If the gravitational field is described by the Kerr metric, show that this metric can be written in the form

$$ds^2 = \left(g_{00} - \frac{g_{03}^2}{g_{33}}\right) c^2 dt^2 + g_{33} \left(d\phi - \Omega dt\right)^2 ,$$

where

$$\Omega = -\frac{g_{03}}{g_{33}} = \frac{r_g a}{(r^2 + a^2) r + r_g a^2} .$$

Use the Equivalence Principle to show that the corresponding non-inertial reference frame rotates with angular velocity Ω.

(b) Find the values of r corresponding to the limit of stationarity ($g_{00} = 0$) and the event horizon ($g_{11} = \infty$).

SECTION B

Each question carries 40 marks. Only marks for the best ONE question will be counted.

1. (a) Consider the motion of a particle in the gravitational field of a Schwarzschild black hole. Using the Hamilton-Jacobi equation, show that

$$E\left(1-\frac{r_g}{r}\right)^{-1}\frac{dr}{dt} = c\sqrt{E^2 - U_{eff}^2},$$

where U_{eff} is the "effective potential energy":

$$U_{eff}(r) = mc^2\sqrt{\left(1-\frac{r_g}{r}\right)\left(1+\frac{L^2}{m^2c^2r^2}\right)},$$

where L is the angular momentum and m is the mass of a particle.

(b) Explain how U_{eff} can be used to find stable and unstable circular orbits.

(c) Show that the radius of the stable circular orbit with angular momentum L is

$$r = \frac{L^2}{m^2c^2r_g}\left[1+\sqrt{1-\frac{3m^2c^2r_g^2}{L^2}}\right].$$

Evaluate the radius of the innermost stable circular orbit.

2. (a) Consider a ring of test particles initially at rest in the (y,z)-plane, perturbed by a plane monochromatic gravitational wave propagating in the x-direction with frequency ω and amplitude h_0. Explain what is meant by "+" and "×" polarizations. Sketch the shape of the ring at times $t = 0$, $\frac{\pi}{2\omega}$, $\frac{\pi}{\omega}$, $\frac{3\pi}{2\omega}$ and $\frac{2\pi}{\omega}$ for two different polarizations of the gravitational wave: (i) $h_+ = h_0 \sin\omega(t - x/c)$, $h_\times = 0$; and (ii) $h_+ = 0$, $h_\times = h_0 \sin\omega(t - x/c)$.

(b) A neutron star of mass m moves around a black hole of mass $M \gg m$ on a circular orbit with radius r. The system emits gravitational radiation with amplitude h and frequency ω. Use the quadrupole formula to show that $\omega = 2\omega_0$, where $\omega_0 = 2\pi/T$ and T is the orbital period. Also show that to an order of magnitude

$$h \sim \frac{m}{M}\frac{r_g^2}{rR},$$

where r_g is the gravitational radius of the black hole and R is its distance.

[This question continues overleaf . . .]

(c) By using the Newtonian approximation for energy show that the radius of the orbit evolves according to

$$r = r_0 \left(1 - \frac{t}{\tau}\right)^{1/4},$$

where r_0 is the initial radius of the circular orbit and

$$\tau \sim \frac{r_g}{2c} \left(\frac{r_0}{r_g}\right)^4 \frac{M}{m}.$$

(d) Assume that such a binary is situated in the Andromeda galaxy ($R \approx 0.7$ Mpc). Estimate to an order of magnitude h and τ in years, if $M = 10^4\ M_\odot$, $m = M_\odot$ and $r_0 = 30\ r_g$.

[End of examination paper.]