MATHEMATICAL TECHNIQUES 3

LECTURE NOTE SET 2

Solving 1st order and 2nd order Differential Equations

Weeks 4-~~5~~6

Solving first and Second Order Linear Differential Equations

In physics we very often deal with quantities which are smoothly varying functions of space and time. It is not surprising then that physical laws are often expressed as equations involving various derivatives of a quantity with respect to position and / or time. Such equations are referred to as Differential Equations and are to be contrasted with purely 'algebraic' equations which do not involve derivatives.

Ex: (i) $m\dfrac{d^2x(t)}{dt^2} = F(x(t))$ ("F = ma")

(ii) $\dfrac{dy}{dx}(x) + 2y(x) + 3 = 0$

The order of a given differential equation is the order of the highest derivative present. In the above (i) is a second-order diff^l equation, whilst (ii) is a first-order equation.

D.E's can be either ordinary or partial.

O.D.E's are those where the unknown function only depends on a single variable, eg $\frac{dy(t)}{dt} + 2t = 4$.

or $\frac{d^2x}{dt^2} = 3t$. Partial Differential Equations (P.D.E) are differential equations where the unknown function depends on more than one variable:-

e.g. Laplace's Equation $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

$(V = V(x,y,z))$.

Schroedinger Equation (time independent)

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right) + V(x,y,z)\psi = E\psi$$

Wave-Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0.$$

Diffusion Equation:

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = \frac{1}{a^2}\frac{\partial P}{\partial t}$$

Finally a Differential Equation (O.D.E or P.D.E) can be classified as either 'linear' or 'non-linear'.

Example of a general linear, 1st order ODE:-

$$\frac{dy(x)}{dx} + P(x)\,y(x) = Q(x).$$

$P(x)$, $Q(x)$ any functions of x only. Note that highest power of y (or $\frac{dy}{dx}$) is 1 – which is why it is a linear ODE.

Example of a non-linear differential equation:-

$$\frac{dy}{dx} = \sin(y(x))$$

non-linear function of $y(x)$.

There are no general rules for solving non-linear differential equations (as opposed to linear differential equations), so we will not consider them in any great detail.

First - Order ODE's

These can be generally written as:-

$$\frac{dy}{dx} = P(x,y)$$

1st order as only involves $\frac{dy}{dx}$. (not $\frac{d^2y}{dx^2}$ etc)

for some function P depending on x and $y(x)$.

Lets look at some simplifying cases for $P(x,y)$:-

$P(x,y) = f(x)\,g(y)$ - factorizable form.

Here f is any function of x only, and g only a function of y but not explicitly x.

$$\frac{dy}{dx} = f(x)\,g(y)$$

$$\therefore \quad \frac{dy}{g(y)} = f(x)\,dx$$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx.$$

Thus if we can calculate the indefinite integrals

$$\int \frac{dy}{g(y)} = G(y) \quad ; \quad \int f(x)\,dx = F(x)$$

for known functions $G(y)$ and $F(x)$,

Then $$G(y) = F(x) + c$$
$$y = G^{-1}(F(x) + c)$$

Note this method can cope even with non-linear 1^{st} order ODE's as long as we can compute the 2 integrals yielding $G(y)$ and $F(x)$.

Ex: $P(x,y) = x\,y^2$. ; $\frac{dy}{dx} = x\,y^2$.

$$G(y) = \int \frac{dy}{y^2} \quad ; \quad F(x) = \int x\,dx$$

$$G(y) = \left(-\frac{1}{y}\right) \qquad F(x) = \frac{1}{2}x^2 + c$$ ← constant of integration.

$$-\frac{1}{y(x)} = \frac{1}{2}x^2 + c \qquad \therefore \quad y(x) = \left[\frac{-1}{\frac{1}{2}x^2 + c}\right]$$

Linear, 1^{st} order ODE's

for convenience

$$P(x,y) = -f(x)y + g(x);$$

$$\boxed{\frac{dy}{dx} + f(x)\,y = g(x)}$$

for any fctns $f(x)$, $g(x)$.

Trick to solving this equation for general $f(x)$, $g(x)$. Let's introduce some function $\alpha(x)$, and multiply our linear ODE with it:-

$$\alpha(x)\frac{dy}{dx} + \alpha(x) f(x)\, y(x) = \alpha(x)\, g(x).$$

Trick is to require that L.H.S above be a total derivative:-

$$\alpha(x)\frac{dy}{dx} + \alpha(x) f(x)\, y(x) \equiv \frac{d}{dx}\left(\alpha(x)\, y(x)\right)$$

$\therefore$ since $\frac{d}{dx}\left(\alpha(x)\, y(x)\right) = \left(\frac{d\alpha}{dx}\right) y + \alpha \frac{dy}{dx}$

we require $\alpha(x)$ to satisfy it's own diff^l equation:-

$$\cancel{y}\left(\frac{d\alpha}{dx}\right) = \alpha(x) f(x)\, \cancel{y(x)}$$

$$\frac{d\alpha}{dx} = \alpha(x) f(x).$$

But we can solve this (simplicity)

$$\int \frac{d\alpha}{\alpha} = \int f(x)\,dx$$

$$\ln \alpha = \int f(x)\,dx + c$$

$$\underline{\alpha(x) = C\, e^{\int f(x)\,dx}}$$

C = constant of integration

Thus, with $\alpha(x)$ defined this way, our linear, 1st order ODE becomes:-

$$\frac{d}{dx}\left(\alpha(x)\,y(x)\right) = \alpha(x)\,g(x).$$

$$\therefore \underbrace{\int \frac{d}{dx}\left(y(x)\,\alpha(x)\right)}_{y(x)\,\alpha(x)} = \int \alpha(x)\,g(x)\,dx.$$

$$y(x)\,\alpha(x) = \int \alpha(x)\,g(x)\,dx + C_2$$

integration constant

So finally, $y(x) = \dfrac{1}{\alpha(x)}\left[\int \alpha(x)\,g(x)\,dx + C_2\right]$

Substituting our expression for $\alpha(x)$ found earlier:-

$$\boxed{y(x) = e^{-\int f(x)dx} \times \left\{\int e^{\int f(x)dx}\,g(x)\,dx + C\right\}}$$

where we have combined all integration constants into C

Note all integrals are <u>indefinite</u>

The form of $y(x)$ is :-

$$y(x) = y_p(x) + y_0(x)$$

where $y_0(x)$ is a solution of our ODE with $g(x) = 0$

that is $y_0(x)$ satisfies:-

$$\frac{dy_0}{dx}(x) + f(x)\,y_0(x) = 0.$$

$$y_0(x) = \text{const} \times e^{-\int f(x)dx}.$$

and $y_p(x)$ is a particular solution depending on the choice of $g(x)$:-

$$y_p(x) = e^{-\int f(x)dx} \times \left[\int e^{\int f(x)dx} g(x)\,dx\right]$$

Sometimes $y_0(x)$ is called the 'complementary function'

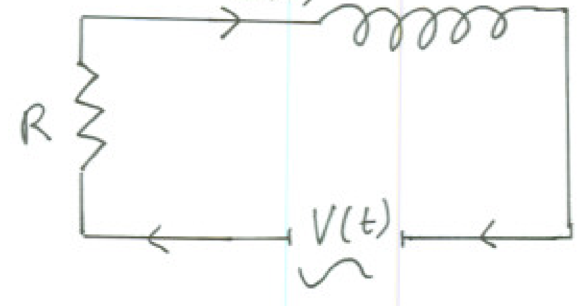
The ODE with $g(x) = 0$:-

$$\frac{dy}{dx} + f(x)\,y = 0$$

is called a homogeneous diff^l equation, as each term contains same power of $y(x)$.

R-L circuit

Example:



Applying kirchoff's current / voltage laws gives:-

$$L\frac{dI(t)}{dt} + RI(t) = V(t)$$

- so this is a linear 1st order ODE for $I(t)$.

Our function $\quad \alpha(t) = e^{\int \frac{R}{L} dt} = e^{Rt/L}$

The general solution is:-

$$I(t) = e^{-Rt/L}\left[\int e^{Rt/L}\frac{V(t)}{L}\,dt + c\right].$$

Special case: what if we choose $V(t) = V_0$ = constant voltage?

$$I(t) = e^{-Rt/L}\left[\int e^{Rt/L}\frac{V_0}{L}\,dt + c\right]$$

$$I(t) = ce^{-Rt/L} + V_0/R.$$

The integration constant C can be given in terms of initial value of the current I at $t=0$:-

$$I(t=0) \equiv I_0 = C + \frac{V_0}{R} \qquad C = \left(I_0 - \frac{V_0}{R}\right)$$

$$I(t) = \left(-V_0/R + I_0\right)e^{-Rt/L} + \frac{V_0}{R}.$$

This example has shown us an important general feature. That a first order ODE has one unknown constant (constant of 'integration') in the expression for the general solution. This constant depends on so called "initial" values or 'boundary values' of our solution at a particular value of t.

So C was determined by $I(t=0)$. Infact if we know I at any given time t_0, this is enough to fix C, because C is a constant and does not change with time.

Ex2:

The equation of motion for a body of mass m, falling under the influence of a constant gravitational field, but with resistive 'drag' present is:-

$$m\frac{dv}{dt} = mg - \beta v$$

g = acceleration due to gravity = constant and β is the 'drag' coefficient. Here we can think of v is the vertical component of the objects velocity as it falls through e.g. Earth's atmosphere.

$$\frac{dV(t)}{dt} = g - \beta/_m V.$$

or $$\frac{dV}{dt} + \beta/_m V = g.$$

So $\alpha(t) = e^{\int \beta/_m \, dt} = e^{\beta/_m t}$

The general solution is :-

$$V(t) = e^{-\beta/_m t}\left(\int e^{\beta/_m t} g \, dt + c\right)$$

$$= e^{-\beta/_m t}\left(\frac{mg}{\beta} e^{\beta/_m t} + c\right)$$

(since g is a constant)

$$V(t) = \frac{mg}{\beta} + c\, e^{-\beta/_m t}$$

If we take $t=0$, $V(0) = V_0$ = initial velocity

$$= \frac{mg}{\beta} + c$$

$$c = V_0 - \frac{mg}{\beta}$$

$$\boxed{V(t) = \frac{mg}{\beta} + \left(V_0 - \frac{mg}{\beta}\right) e^{-\beta/_m t}}$$

notice that if $V_0 = \frac{mg}{\beta}$, then $V(t)$ = constant!
physically the forces on the falling object are zero in this case: the acceleration due to gravity is balanced by the friction due to air.

Solving 2nd order ODE's

The most general linear, 2nd order ODE can be written as:-

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = F(x)$$

where P, Q and F are functions of x only.

When $F=0$, the above is a homogeneous 2nd order ODE.

We will discuss the inhomogeneous case ($F \neq 0$) later.

For now, lets concentrate on the equation:-

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$$

Some Simplifying Special Cases:-

(i) If $Q(x)=0$, then $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} = 0$.

If we call $u(x) = \frac{dy}{dx}$, then $\frac{du}{dx} + P(x)u(x) = 0$

This is a homogeneous 1st order ODE - which we can solve e.g. by methods discussed earlier. Even if we take $F(x) \neq 0$ we can still solve $\frac{du}{dx} + P(x)u(x) = F(x)$

Once an exact solution is known, we finally have to solve the Diff^{l} Equation:-

$$\frac{dy(x)}{dx} = u(x) . \qquad , \quad y(x) = \int u(x)dx + c_2$$

to get $y(x)$.

Since the solution for $u(x)$ will involve some unknown constant of integration, call it c_1 (see last section), we can see that $y(x)$ depends on 2 arbitrary constants, c_1 and c_2. In the previous section we found that solutions of 1st order linear ODE depend only on <u>one</u> arbitrary constant. Both c_1 and c_2 can be fixed by <u>bondary conditions</u>

<u>Ex:</u> The equation of motion for a body of mass m moving under the force of friction depending linearly on it velocity is:-

$$m\frac{d^2y}{dt^2} - \beta\frac{dy}{dt} = 0$$

(here for simplicity we only consider motion in 1-direction)

So let $u(t) = \frac{dy}{dt}$

$$m\frac{du}{dt} - \beta u = 0$$

Solving:- $\int \frac{du}{u} = \int \beta/m \, dt = \beta/m \, t + c_1$

$$\ln(u) = \beta/m \, t + c \implies u(t) = e^{\beta/m \, t + c_1}$$

Finally, we have to solve:

$$u(t) = \frac{dy}{dt} = e^{\beta/m\, t + c_1} \quad \text{to obtain } y(t).$$

$$\int dy = \int e^{\beta/m\, t + c_1}\, dt \Rightarrow \underline{y(t) = \left(\frac{m}{\beta} e^{\beta/m\, t + c_1} + c_2\right)}$$

Note that $y(t=0) = u(t=0) = \frac{m}{\beta} e^{c_1} + c_2$.

and $\left.\frac{dy}{dt}\right|_{t=0} = u(t=0) = e^{c_1}$

So by specifying the values of $y(t=0)$, and $\left.\frac{dy}{dt}\right|_{t=0}$

c_1 and c_2 are determined.

(ii) $\underline{P = Q = 0, \quad F \neq 0}$.

$$\frac{d^2y}{dx^2} = F(x). \quad \text{Integrating directly:-}$$

$$\frac{dy}{dx} = \int \frac{d^2y}{dx^2}\, dx = \int F(x)\, dx$$

$$\Rightarrow \quad y(x) = \int \frac{dy}{dx}\, dx = \int \left(\int F(x) dx\right) dx.$$

As long as we can evaluate $\int F(x) dx$ and $\int \left(\int F(x)\, dx\right) dx$

$y(x)$ is obtained.

More General cases : Series Solutions.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0 .$$

Lets try and find a series solution of form:-

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad , \; a_n \text{ - coefficients to be determined.}$$

This method could have been used to solve the general 1st order ODE of last section, if all other methods fail. But the series method depends on being able to solve for a_n, which is not always exactly possible.

The idea is to substitute our series expansion into the ODE and obtain a (hopefully) solvable set of algebraic equations. For general functions $P(x)$, $Q(x)$ this is difficult to carry out. But lets look at a familiar example to see how this method works.

Consider case $P = 0$, $Q = \omega^2 > 0$, ω a constant.

$$\frac{d^2y}{dx^2} = -\omega^2 y .$$

if '$x = t$' this is related to the familiar equation of motion of a simple harmonic oscillator

We know the general solution is simply:-

$$y(x) = A\sin(\omega x) + B\cos(\omega x)$$

where A, B are the familiar integration constants (2 because y satisfies a 2^{nd} order ODE).

But lets see how the series method can reproduce the above solution.

If $\quad y(x) = \sum_{n=0}^{\infty} a_n x^n; \qquad \frac{dy}{dx} = \sum_{n=0}^{\infty} n a_n x^{n-1}$

and $\quad \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}.$

$$\therefore \quad \underbrace{\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}}_{\frac{d^2y}{dx^2}} = \underbrace{-\omega^2 \sum_{n=0}^{\infty} a_n x^n}_{-\omega^2 y(x)}$$

now on L.H.S, there is no contribution from $n=0, n=1$ terms. Therefore:-

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -\omega^2 \sum_{n=0}^{\infty} a_n x^n$$

let $n' = n-2$,

$$\sum_{n'=0}^{\infty} (n'+2)(n'+1) a_{n'+2} x^{n'} = -\omega^2 \sum_{n=0}^{\infty} a_n x^n$$

i.e. $$\sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2}x^n + \omega^2 a_n x^n\right) = 0$$

$$= \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + \omega^2 a_n\right]x^n = 0$$

The vanishing of an infinite power series, for any value of x requires all it's coefficients to vanish.

So we find :-

$$(n+2)(n+1)a_{n+2} + \omega^2 a_n = 0 \quad , n = 0, 1, 2, \dots$$

This equation defines a so-called recursion-relation for the a_n. Putting in values for $n = 0, 1, 2, \dots$ we find :-

$n=0.$ $\quad 2.1.a_2 = -\omega^2 a_0$

$n=1$ $\quad 3.2.a_3 = -\omega^2 a_1$

$n=2$ $\quad 4\cdot 3\cdot a_4 = -\omega^2 a_2 = \frac{1}{2}(-\omega^2)^2 a_0$

$n=3$ $\quad 5\cdot 4\cdot a_5 = -\omega^2 a_3 = \frac{1}{6}(-\omega^2)^2 a_1$

$\vdots$ etc.

So <u>all</u> the a_n for n odd are related to a_1 and all the a_n for n even are related to a_0 !

The solution of the recursion relations for a_n (see Maple Ex. classes)

$$\begin{cases} a_n = \dfrac{(-1)^{\frac{n-1}{2}}\omega^{2n-1}}{(2n+1)!}\, a_1 & n = 1, 3, 5, \ldots \\ a_n = (-1)^{\frac{n}{2}}\dfrac{\omega^{n}}{n!}\, a_0 & n = 0, 2, 4, \ldots \end{cases}$$

So our solution is :-

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \underbrace{\sum_{n=0,2,4\ldots}^{\infty} a_n x^n}_{\text{sum over } n \text{ odd}} + \underbrace{\sum_{n=1,3,5\ldots}^{\infty} a_n x^n}_{\text{sum over } n\text{-even}}$$

$$y(x) = \sum_{n \text{ odd}} \frac{(-1)^{\frac{n-1}{2}}\omega^{2n-1}}{(2n+1)!} a_1 x^n + \sum_{n \text{ even}} \frac{(-1)^{n/2}\omega^n a_0 x^n}{n!}$$

$$= a_1 \sum_{n \text{ odd}} \frac{(-1)^{\frac{n-1}{2}}\omega^{2n} x^n}{(2n+1)!} + a_0 \sum_{n \text{ even}} \frac{(-1)^{\frac{n}{2}}\omega^n x^n}{n!}$$

now recall the series expansion for $\sin \omega x$ and $\cos \omega x$:-

$$\sin(\omega x) = \sum_{n \text{ odd}} \frac{(-1)^{\frac{n-1}{2}}(\omega x)^n}{n!}$$

$$\cos(\omega x) = \sum_{n \text{ even}} \frac{(-1)^{n/2}(\omega x)^n}{n!}$$

Therefore we have fond our solution:-

$$y(x) = a_1 \sin(\omega x) + a_0 \cos(\omega x)$$

where a_1, a_0 are arbitrary constant and play the role of constants of integration.

Note also, we can define a slighly more general series expansions than we have used so far,

$$y(x) = \sum_{n=0}^{\infty} a_n x^{k+n}$$

where k could be any real number (not necessary integer valued). In our Simple Harmonic example we took $k=0$ and still obtained most general solution. For other 2nd order ODE's, we may have to take $k \neq 0$.

Ex2: Hermite's Diff^l equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2\alpha\, y(x) = 0$$

α some real number.

- This is clearly a 2nd order, linear (and homogeneous) ODE.

A trick to simplify the problem of solving this equation is to introduce a new function $u(x)$, related to $y(x)$ by:-

$$y(x) = e^{x^2/2} u(x).$$

Then $$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x^2/2} u(x)\right) = x e^{x^2/2} u(x) + e^{x^2/2} \frac{du}{dx}.$$

$$\frac{d^2y}{dx^2} = x^2 e^{x^2/2} u(x) + e^{x^2/2} u(x) + x e^{x^2/2} \frac{du}{dx} + x e^{x^2/2} \frac{du}{dx} + e^{x^2/2} \frac{d^2u}{dx^2}$$

$$= (x^2+1) e^{x^2/2} u(x) + 2x e^{x^2/2} \frac{du}{dx} + e^{x^2/2} \frac{d^2u}{dx^2}.$$

Hence Hermite's DE becomes:-

$$(\cancel{x^2}+1) e^{x^2/2} u(x) + \cancel{2x e^{x^2/2} \frac{du}{dx}} + e^{x^2/2} \frac{d^2u}{dx^2} - 2x\left(\cancel{x e^{x^2/2} u} + \cancel{e^{x^2/2} \frac{du}{dx}}\right) + 2\alpha e^{x^2/2} u = 0$$

$$e^{x^2/2}\left[\frac{d^2u}{dx^2} + (2\alpha + 1 - x^2)u\right] = 0$$

i.e. $$\boxed{\frac{d^2u}{dx^2} + (2\alpha+1-x^2)u = 0}$$

So if we can solve this equation for $U(x)$ we have our solution as $y(x) = e^{x^2/2} U(x)$.

This last equation has important apps in Physics, in particular it is related to the Schroedinger equation for the <u>Quantum Mechanical simple harmonic oscillator</u> (see QMA / QMB).

Schrodinger Eqn: $\hat{H}\,\psi(x) = E\,\psi(x)$.

$\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$. For a particle moving in a simple harmonic potential, $V(x) = \frac{1}{2}m\omega^2 x^2$

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \left(\frac{1}{2}m\omega^2 x^2 - E\right)\psi(x) = 0$$

or
$$\frac{d^2\psi}{dx^2} + \left(\frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2}x^2\right)\psi = 0$$

To make this look like our equation for $U(x)$,

let $x = \sqrt{\frac{\hbar}{m\omega}}\, z$ (rescaled co-ord)

$$\frac{d^2\psi}{dz^2} + (\lambda - \underset{=z^2}{\cancel{x^2}})\psi = 0 \quad , \quad \lambda = \frac{2E}{\hbar\omega}$$

So $\psi(x) = u(x)$ if we identify $\lambda = 2\alpha + 1$.

Therefore finding a solution to Hermite's D.E. $y(x)$, allows us to obtain wave-function of the Q.M. harmonic oscillator $\psi(x) = u(x) = e^{-x^2/2}\, y(x)$!

So let's try and find a series solution :-

$$y(x) = \sum_{n=0}^{\infty} a_n x^{k+n}$$

Substituting this into the Hermite DE, we find the recursion relation [See Homework questions] :-

$$k(k-1) = 0 .$$

$\boxed{k=0}$;

$$a_{j+2} = 2a_j \frac{(j-\alpha)}{(j+1)(j+2)} \qquad j = 0, 2, 4, \ldots$$

and for $\boxed{k=1}$,

$$a_{j+2} = \frac{2a_j\,(j+1-\alpha)}{(j+2)(j+3)} \qquad j = 0, 2, 4, \ldots$$

[The solutions of these recursion relations will be obtained as part of the Maple Ex. classes].

So for $\boxed{k=0}$ we find the series solution:-

$$y_{even}(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 \left[1 + \frac{2(-\alpha)x^2}{2!} + \frac{2^2(-\alpha)(2-\alpha)x^4}{4!} + \cdots \right]$$

and for $\boxed{k=1}$ we find the series solution:-

$$y_{odd}(x) = \sum_{n=0}^{\infty} a_n x^{\overset{k}{\overset{\shortparallel}{1}}+n} = a_0 \left[x + \frac{2(1-\alpha)x^3}{3!} + \frac{2^2(1-\alpha)(3-\alpha)x^5}{5!} + \cdots \right]$$

Note that the 'even', 'odd' subscripts above indicate the solution is either and even or odd function as $x \to -x$.

Special case of α = integer.

For general real value of the parameter α, both of the solutions above, ($y_{even}(x)$, $y_{odd}(x)$) are infinite power series in x. For the special case when $\alpha = n$, n some positive integer a dramatic simplification happens: the series stop after a finite number of terms!

for example: take $\alpha = n = 0$, then for $k=0$,

$$y_{even}(x) = a_0 = \text{constant}.$$

$n=1$, $\quad y_{odd}(x) = a_0 x$

$n=2$, $\quad y_{even}(x) = a_0 - a_0 2x^2 = a_0(1-2x^2)$

$n=3$ $\quad y_{odd}(x) = a_0 [x - \frac{2}{3}x^3]$

⋮ etc..

Infact, it turns out that we can write our

solution as:- $\quad y_{(n)}(x) = H$

$$y_{(n=0)}(x) = a_0 = a_0 H_0(x).$$

$$y_{(n=1)}(x) = a_0 x = \frac{1}{2} a_0 H_1(x)$$

$$y_{(n=2)}(x) = a_0(1-2x^2) = -\frac{1}{2} a_0 H_2(x)$$

$$y_{(n=3)}(x) = a_0(x - \tfrac{2}{3}x^3) = -\frac{1}{12} a_0 H_3(x)$$

⋮ etc.

The functions $H_0(x)$, $H_1(x)$, $H_2(x)$, ...

are the so called Hermite Polynomials:-

$$H_0 = 1 \qquad H_3 = 8x^3 - 12x$$

$$H_1 = 2x \qquad \vdots$$

$$H_2 = 4x^2 - 2 \qquad \text{etc.}$$

So we see that the special case $\boxed{\alpha = n}$,

Hermite's equation is solved by simple

polynomial functions!