Queen Mary
UNIVERSITY OF LONDON

M.Sc. Astrophysics

ASTM112 Astrophysical Fluid Dynamics

Duration: 3 hours
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You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best **3** *questions answered will be counted.*

Calculators ARE permitted in this examination.

Notation

The following notation is used throughout unless otherwise stated. The pressure, density, gravitational potential and adiabatic exponents are denoted by p, ρ, ψ, Γ_1 and Γ_3 respectively. The equilibrium values of these quantities are sometimes distinguished using a zero subscript. The position vector is denoted by $\mathbf{r}$ or $\mathbf{x}$, the time by t, the velocity by $\mathbf{u}$, the surface radius of a spherical configuration by R, and the gravitational constant by G. Vectors are denoted by boldface type.

Astronomical and Physical Data

Mass of the Sun	$M_\odot$	$2.0 \times 10^{30}\,\mathrm{kg}$
Surface radius of the Sun	$R_\odot$	$7.0 \times 10^{8}\,\mathrm{m}$
Luminosity of the Sun	$L_\odot$	$3.8 \times 10^{26}\,\mathrm{J\,s^{-1}}$
Gravitational constant	G	$6.67 \times 10^{-11}\,\mathrm{kg^{-1}m^3s^{-2}}$
Speed of light in a vacuum	c	$3.0 \times 10^{8}\,\mathrm{m\,s^{-1}}$

Standard Formulae

Candidates may assume the following set of basic equations and formulae:

In spherical polar coordinates (r, θ, ϕ)

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}, \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right)$$

and

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\,.$$

For $\mathbf{u} = (u_r, u_\theta, u_\phi)$,

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}$$

 [Turn page]

and

$$\nabla\times\mathbf{u} \;=\; \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ u_r & ru_\theta & r\sin\theta\, u_\phi \end{vmatrix} .$$

The spherical harmonic $Y_l^m(\theta,\phi) = P_l^{|m|}(\cos\theta)\exp(im\phi)$, where $P_l^{|m|}$ denotes the associated Legendre function, satisfies the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_l^m}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_l^m}{\partial\phi^2} + l(l+1)Y_l^m \;=\; 0\,,$$

where l is a non-negative integer and m is an integer such that $|m| \le l$. Further

$$\nabla^2(Y_l^m r^l) \;=\; 0 \qquad \nabla^2(Y_l^m r^{-l-1}) \;=\; 0\,.$$

In cylindrical polar coordinates (r,ϕ,z), with $\mathbf{u} = (u_r, u_\phi, u_z)$,

$$\nabla\cdot\mathbf{u} \;=\; \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z}\,, \qquad \nabla\times\mathbf{u} \;=\; \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & \mathbf{e}_z \\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z \\ u_r & ru_\phi & u_z \end{vmatrix} .$$

The material derivative is given by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}.\nabla\,.$$

The equation of motion for an inviscid fluid may be assumed in the form

$$\rho\frac{D\mathbf{u}}{Dt} \;=\; -\nabla p \;-\; \rho\nabla\psi\,.$$

The continuity equation may be assumed in the form

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) \;=\; 0\,.$$

The energy equation may be assumed in the form

$$\frac{Dp}{Dt} - \frac{\Gamma_1 p}{\rho}\frac{D\rho}{Dt} \;=\; \rho(\Gamma_3 - 1)\left(\epsilon - \frac{1}{\rho}\nabla\cdot\mathbf{F}\right)\,,$$

where ϵ is the heat generated per unit mass, and $\mathbf{F}$ is the heat flux. For adiabatic motion, the right-hand side of this equation is zero.

The gravitational potential satisfies Poisson's equation, $\nabla^2\psi \;=\; 4\pi G\rho$, which may be assumed to have the solution

$$\psi(\mathbf{r},t) \;=\; -\int_V \frac{G\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V'\,,$$

where the integration is taken over the fluid volume V, and $\mathrm{d}V'$ denotes the volume element $\mathrm{d}^3\mathbf{r}'$.

[Turn page]

1 A compressible fluid is stratified in uniform gravitational field, with gravitational acceleration g_0 directed downwards. Equilibrium pressure $p_0(z)$ and density $\rho_0(z)$ depend on vertical coordinate z only; the axis z points upwards.

(a) Show that in hydrostatic equilibrium,

$$\frac{dp_0(z)}{dz} = -\rho_0(z)g_0.$$

Explain the physical nature of this hydrostatic relation.

(b) Consider a small vertical displacement of a small fluid element from its equilibrium position at $z = 0$. Assume that the fluid element is always in pressure equilibrium with surrounding fluid, and the surrounding fluid remains undistorted.

Show that in the adiabatic approximation (i.e. discarding any heat exchange between the fluid element and the surrounding fluid), the small variations of pressure δp and density $\delta\rho$ in the fluid element satisfy

$$\delta p = \Gamma_1 \frac{p_0(0)}{\rho_0(0)}\,\delta\rho,$$

where Γ_1 is the adiabatic exponent. Discuss briefly, in which circumstances the adiabatic approximation is applicable for describing astrophysical fluids.

(c) Show that after such a displacement by an amount δz in vertical direction, the density in the fluid element will differ from the density of the fluid which is around it by an amount

$$\rho' = \frac{\rho_0(0)}{g_0} N^2(0)\,\delta z,$$

where

$$N^2 = -g_0\left(\frac{d\ln\rho_0}{dz} - \frac{1}{\Gamma_1}\frac{d\ln p_0}{dz}\right).$$

Explain, qualitatively, how the motion of the fluid element will develop after this initial displacement, when (i) $N^2 > 0$ and (ii) $N^2 < 0$.

(d) Show that linearized equation of motion of this particular fluid element can be written as

$$\rho_0(0)\frac{d^2\delta z}{dt^2} = -\rho' g_0 = -\rho_0(0)N^2(0)\,\delta z.$$

Show further that the solution to this equation is $\delta z(t) = A\exp(i\omega t)$, with $\omega^2 = N^2$. When $N^2 > 0$, what is the restoring force of the resulted oscillatory solution? When $N^2 < 0$, you have a solution which grows exponentially with time. Discuss briefly the physical nature of this solution. Why our analysis is not applicable for predicting correctly the development of this solution at large t?

[Turn page]

2 (a) Starting from the equation of motion in a frame rotating uniformly with angular velocity $\mathbf{\Omega}$,

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p - \nabla\psi - 2\mathbf{\Omega}\times\mathbf{u} - \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) ,$$

where $\mathrm{d}/\mathrm{d}t$ denotes the material derivative in the rotating frame, derive the equation for hydrostatic equilibrium of the fluid in the form

$$\frac{1}{\rho}\nabla p = -\nabla\Phi ,$$

where the total effective potential Φ should be defined and its form carefully derived. Explain why, in the equilibrium, pressure p is constant, and also density ρ is constant, at any surface where Φ is constant.

(b) Explain why Φ is constant over the surface of a rotating fluid body. Show further that for a slow rotation, the relative difference $\Delta R/R$ between the equatorial radius $R + \Delta R$ and the polar radius R can be estimated as

$$\frac{\Delta R}{R} = \frac{1}{2}\frac{\Omega^2 R^3}{GM}.$$

Evaluate the magnitude of this relative difference for Jupiter, which has rotation period 10 hours, radius 0.1 solar radii, and mass 10^{-3} solar masses [you are referred to the solar data given on page 1 of this paper].

Explain, qualitatively, how the last result will be modified if we take into account a distortion of the spherically-symmetric gravitational field induced by the rotational distortion of the mass distribution inside the planet.

(c) A binary star system consists of two stars with masses M_1 and M_2 separated by a distance a, orbiting each other in circular orbits. Show that the period $2\pi/\Omega$ of the system is such that

$$\Omega^2 = \frac{G(M_1 + M_2)}{a^3} .$$

Assuming that the gravitational potential of each star can be approximated by that of a point mass at the centre of the star, and taking Cartesian coordinates such that the x-axis runs through the centres of the two stars and the orbits are in the $z = 0$ plane, derive the Roche form of the potential Φ. Sketch contours of constant Φ in the $z = 0$ plane, and explain the significance of the Roche lobes.

[Turn page]

3 A gaseous configuration moves under its internal pressure and self-gravity. Suppose that $p = p_0 + p'$, $\rho = \rho_0 + \rho'$ and $\psi = \psi_0 + \psi'$, where p', ρ' and ψ' are the perturbations to pressure, density and gravitational potential respectively, and p_0, ρ_0 and ψ_0 are equilibrium quantities. The perturbations and velocity $\mathbf{u}$ are small, so that quadratic and higher order expressions involving them may be neglected. By linearizing the basic fluid equations show that

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p' - \rho' \nabla \psi_0 - \rho_0 \nabla \psi',$$

$$\frac{\partial \rho'}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{u}),$$

$$\nabla^2 \psi' = 4\pi G \rho',$$

and, in the adiabatic approximation,

$$\frac{\partial p'}{\partial t} + \mathbf{u} \cdot \nabla p_0 = \Gamma_1 \frac{p_0}{\rho_0} \left(\frac{\partial \rho'}{\partial t} + \mathbf{u} \cdot \nabla \rho_0 \right).$$

Now prove that the linear equations of adiabatic radial stellar oscillations can be written as

$$\frac{dU}{dr} = \left(\frac{g_0}{c^2} - \frac{2}{r} \right) U - \frac{1}{\rho_0 c^2} p_1,$$

$$\frac{dp_1}{dr} = \left(\omega^2 - N^2 + 4\pi G \rho_0 \right) \rho_0 U - \frac{g_0}{c^2} p_1,$$

where $U(r)$ and $p_1(r)$ describe radial displacements $\delta \mathbf{r}$ and Eulerian pressure perturbations p' through

$$\delta \mathbf{r} = \hat{\mathbf{r}} U(r) \exp(i\omega t) \quad \text{and} \quad p' = p_1(r) \exp(i\omega t).$$

You should assume perturbations of the form $\rho' = \rho_1(r) \exp(i\omega t)$ and $\psi' = \psi_1(r) \exp(i\omega t)$. ω is the angular frequency, $\mathbf{u} = i\omega \delta \mathbf{r}$, $\hat{\mathbf{r}}$ is unit vector along r, c is adiabatic sound speed, g_0 is equilibrium gravitational acceleration, and N is Brunt-Vâisâlâ frequency. You may use the relations

$$c^2 = \Gamma_1 \frac{p_0}{\rho_0}, \quad N^2 = -g_0 \left(\frac{d \ln \rho_0}{dr} - \frac{1}{\Gamma_1} \frac{d \ln p_0}{dr} \right).$$

[**Hint:** By comparing the continuity and the Poisson equations, show that $d\psi_1/dr = -4\pi G \rho_0 U$].

Give brief mathematical and physical explanations why these equations are not adequate for describing the excitation and damping of stellar oscillations.

[Turn page]

4 A fluid has motion and variations only in one spatial direction x. By appropriately combining the momentum and continuity equations in their standard form (with no external forces), and the adiabatic energy equation which you may assume in the form

$$\rho\frac{DU}{Dt} = \frac{p}{\rho}\frac{D\rho}{Dt},$$

U being the internal energy per unit mass, derive the momentum and energy equations in conservative form:

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) = 0,$$

$$\frac{\partial}{\partial t}\left(\rho U + \frac{1}{2}\rho u^2\right) + \frac{\partial}{\partial x}\left(\rho u\left(U + \frac{p}{\rho} + \frac{1}{2}u^2\right)\right) = 0.$$

Hence deduce the jump conditions for a steady shock:

$$\begin{aligned} \rho_1 u_1 &= \rho_2 u_2 \\ \rho_1 u_1^2 + p_1 &= \rho_2 u_2^2 + p_2 \\ U_1 + \frac{p_1}{\rho_1} + \frac{1}{2}u_1^2 &= U_2 + \frac{p_2}{\rho_2} + \frac{1}{2}u_2^2 \end{aligned}$$

where subscripts 1 and 2 denote conditions just upstream and just downstream of the shock.

For an ideal gas, $U = (\Gamma - 1)^{-1}p/\rho$, where Γ is the adiabatic exponent. Show that for a strong shock, for which the upstream Mach number $M_1 \gg 1$, the jump conditions imply

$$\begin{aligned} \frac{\rho_2}{\rho_1} &= \frac{\Gamma+1}{\Gamma-1} = \frac{u_1}{u_2} \\ \frac{p_2}{p_1} &= \frac{2\Gamma M_1^2}{\Gamma+1}. \end{aligned}$$

Consider the following highly simplified model based on the above. The Sun loses mass at a rate $\dot{M}$ per unit time in the form of the solar wind. At the orbit of the earth the wind is supersonic and the measured velocity is u_E. At a greater distance r_s from the Sun, the wind encounters a strong stationary shock. At an even greater distance, r_h, the wind encounters the heliopause, the boundary between the solar wind and the interstellar medium (which are assumed not to interpenetrate). Assume that the speed in the wind is constant in the supersonic regime, and $\Gamma = 5/3$. Deduce that the shock is located at

$$r_s = \left(\frac{\dot{M}}{\pi u_E \rho_2}\right)^{1/2}.$$

Between the shock and the heliopause, ρ and $\rho u^2 + p$ are essentially constant, and the speed u declines rapidly with distance from the Sun, so that $\rho u^2 + p = p_x$, where p_x is the pressure in the interstellar medium. Show that

$$r_s = \left(\frac{\dot{M}u_E}{4\pi p_x}\right)^{1/2}.$$

Assuming that $\dot{M}$ is constant over the Sun's main-sequence lifetime, show that the location of the heliopause varies slowly with time as

$$r_h \propto t^{1/3}.$$

Taking $\dot{M} = 10^{-13}M_\odot$ per year, $u_E = 4\times10^5\ \mathrm{m\,s^{-1}}$, the age of the Sun to be 5×10^9 years, and p_x to be $10^{-14}\ \mathrm{N\,m^{-2}}$ (appropriate to a temperature of 10^4K and a density of 10^5 atoms $\mathrm{m^{-3}}$), estimate the present positions of the shock and the heliopause.

[Turn page]

5 Write briefly on **two** of the following topics:

(a) differential rotation and meridional circulation in stars;

(b) solar seismology;

(c) formation of supersonic flows in compressible fluids;

(d) nonlinear acoustic waves and shocks.

[End of paper]
S. V. Vorontsov
A. G. Polnarev