Queen Mary
UNIVERSITY OF LONDON

M.Sc. EXAMINATION

ASTM051 Plasma Astrophysics
MTHMN82 Astrophysical Fluids and Plasmas [Plasmas Section only]

Wednesday, 12 June 2002
18:30-20:00

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each section.

Calculators are NOT permitted in this examination. Numerical answers where required may be determined approximately, to within factors ~ 5.

You may quote the following results unless the question specifically asks you to derive it. All notation is standard. Vectors are denoted by boldface type, e.g., $\mathbf{A}$ *while scalars, including the magnitude of a vector, are in italics, e.g.,* $|\mathbf{E}| = E$.

(i) The Lorentz force on a particle of charge q moving in electric and magnetic fields $\mathbf{E}$ *and* $\mathbf{B}$ *respectively is given by*

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

(ii) Maxwell's Equations

$$\begin{aligned}
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \\
\nabla \cdot \mathbf{E} &= \frac{\rho_q}{\varepsilon_0} \\
\nabla \cdot \mathbf{B} &= 0
\end{aligned}$$

where $\mu_0 \varepsilon_0 = 1/c^2$.

(iii) The MHD equations for a plasma with electrical conductivity σ*:*

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) &= 0 \\
\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} &= -\nabla P + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} \\
\left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \left(P \rho^{-\gamma} \right) &= 0 \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B} \\
\mathbf{E} + \mathbf{V} \times \mathbf{B} &= \mathbf{j}/\sigma
\end{aligned}$$

(iv) Conservation forms of the ideal MHD momentum and energy equations

$$\frac{\partial(\rho\mathbf{V})}{\partial t} + \nabla\cdot\left[\rho\mathbf{V}\mathbf{V} + \left(P + \frac{B^2}{2\mu_0}\right)\boldsymbol{I} - \frac{\mathbf{B}\mathbf{B}}{\mu_0}\right] = \mathbf{0}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho V^2 + \frac{P}{\gamma-1} + \frac{B^2}{2\mu_0}\right) + \nabla\cdot\left(\mathbf{V}\frac{1}{2}\rho V^2 + \frac{\gamma}{\gamma-1}P\mathbf{V} + \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}\right) = 0$$

(v) The following vector identities and relations

$$\begin{aligned}
\nabla\times(\mathbf{a}\times\mathbf{b}) &= \mathbf{a}(\nabla\cdot\mathbf{b}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - \mathbf{b}(\nabla\cdot\mathbf{a}) - (\mathbf{a}\cdot\nabla)\mathbf{b} \\
(\nabla\times\mathbf{B})\times\mathbf{B} &= (\mathbf{B}\cdot\nabla)\mathbf{B} - \nabla\left(\frac{B^2}{2}\right) \\
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \\
(\mathbf{a}\times\mathbf{b})\times\mathbf{c} &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}
\end{aligned}$$

(vi) The following numerical values of physical constants and parameter values:

Name	symbol	value
Electronic Charge	e	1.6×10^{-19} C
Electron volt	eV	1.6×10^{-19} Joules
Electron mass	m_e	9.1×10^{-31} kg
Proton mass	m_p	1.67×10^{-27} kg
Permeability of free space	μ_0	$4\pi\times10^{-7}$ Henry/m
Permittivity of free space	ε_0	8.85×10^{-12} Farad/m
Speed of light in vacuo	c	3×10^{8} m/s
Earth Radius	R_e	6371 km
Astronomical Unit	AU	1.5×10^{11} m

SECTION A

Each question carries 10 marks. You should attempt ALL questions.

1. Describe in one or two sentences what is meant by each of the following terms:

(a) Debye Shielding

(b) Cyclotron motion

(c) Magnetic moment conservation

2. A particle of mass m and charge q moves non-relativistically in a uniform magnetic field $\mathbf{B}_0 = B_0\hat{z}$ together with a uniform electric field $\mathbf{E}_0 = E_0\hat{y}$. Given that when $E_0 = 0$ the motion in the $x-y$ plane is circular, show that in the case $E_0 > 0$ all particles also drift with a speed E_0/B_0 in the $+\hat{x}$ direction.

3. Explain what is meant by the critical radius in Parker's solar wind model. Sketch the families of possible solutions to the solar wind equations (i.e., radially symmetric isothermal steady outflow) and discuss briefly the boundary conditions which determine the physically correct choice(s).

4. Draw a sketch of the magnetic reconnection configuration proposed by Petschek. On your sketch show the magnetic field lines, plasma flow, separatrices, slow shocks, and diffusion region. Describe briefly an astrophysical situation in which reconnection is believed to be an important process. Support your description with a suitable illustration.

SECTION B

Each question carries 30 marks. You may attempt all questions but only marks for the best two questions will be counted.

1. A proton of (non-relativistic) energy 1MeV (= 1.6×10^{-13}J) is injected into the Earth's van Allen radiation belts. Initially the proton is at $4R_e$ ($1R_e$ is an Earth radius which you may take to be $\approx$ 6000km) from the Earth's centre and in the equatorial plane, where it has a pitch angle of $45°$. The magnetic field there is approximately 5.0×10^{-7}Teslas.

(a) [8 marks] Calculate roughly the gyroperiod and Larmor radius of the proton, and hence deduce that Larmor radius is small compared to the scale of variation of the Earth's magnetic field (which you may take to be a dipole, with dependence $\propto 1/r^3$).

(b) [10 marks] Estimate the drift velocity of the proton due to the curvature drift near the equatorial region. [You will need to derive an expression for this drift; you may use the result for the $\mathbf{E} \times \mathbf{B}$ drift found in Section A.] Be sure to indicate the direction of this drift.

(c) [9 marks] Assuming the only variation of the field magnitude is with radial distance, find the minimum radius which the particle can reach due to magnetic mirroring.

(d) [3 marks] Hence describe the long-term motion of this particle.

2. A uniform perfectly conducting spherical plasma cloud of mass density ρ_0, total mass M and radius R_0 is rotating at an angular frequency Ω_0. The cloud is permeated by a magnetic field which is everywhere aligned with the rotation axis, has a magnitude B_0 at the equatorial surface, and increases linearly to twice that value at the rotation axis.

(a) [12 marks] This cloud now suffers a collapse to a radius $R_1 < R_0$. The cloud remains uniform so that its total (conserved) angular momentum throughout the collapse phase is given by $\frac{2}{5}MR_0^2\Omega_0$. Find the resulting mass density, ρ_1, and angular velocity, Ω_1. Hence deduce that, at the equatorial surface the centrifugal force per unit volume, $\rho_1\Omega_1^2R_1$, has increased by a factor $(R_0/R_1)^6$ over its initial value.

(b) [12 marks] Use simple flux freezing arguments (without performing any explicit integrations) to estimate the factor by which the magnetic field, B_1, at the equatorial surface has increased. Hence, assuming that the magnetic profile remained roughly linear, *estimate*, e.g., to within a factor of order unity, the magnetic pressure force per unit volume at the equatorial surface and give an expression for this force in terms of its initial value.

(c) [6 marks] Finally, by comparing the centrifugal and magnetic forces found above, show that the magnetic forces increase relatively less, by a factor R_1/R_0, than the rotational ones, so that the conservation of angular momentum represents a more serious inhibitor of further collapse than the build up of magnetic pressure.

3. A shock wave is propagating in a uniform medium which, outside the shock transition itself, satisfies the MHD equations. The shock normal is $\mathbf{n} = -\hat{x}$. In the frame in which the shock is at rest, the upstream field and fluid parameters are $\rho = \rho_u$, $\mathbf{V} = u_u\hat{x}$, $P = P_u$, $\mathbf{B} = B_u\hat{x}$. A subscript "d" denotes values downstream of the shock.

(a) [8 marks] Show that $\rho_d/\rho_u = u_u/V_{xd} \equiv r$ and that $B_{xd} = B_u$

(b) [8 marks] Show also that if the tangential field or flow components remain zero downstream, i.e., $V_{yd} = 0 = V_{zd}$ and $B_{yd} = 0 = B_{zd}$, the shock jump conditions for transverse momentum and induction are satisfied.

(c) [4 marks] Hence deduce that for such parallel shocks the Rankine-Hugoniot relations reduce to the shock jump conditions in an unmagnetised fluid.

(d) [10 marks] By eliminating the downstream pressure between the x-momentum and energy conservation equations, and using your previous results, show that r is given by

$$r = \frac{\gamma+1}{\dfrac{2}{M_{cs}^2}+\gamma-1}$$

where $M_{cs} \equiv \sqrt{\rho_u u_u^2/\gamma P_u}$ is the sonic Mach number of the incident flow.

 [End of examination paper.]