Answers to 3C26/2004.

1. (Unseen Problem.)

$$\int \mid \Psi \mid^2 d\tau = 1$$

gives

$$N^2 \int_0^\infty e^{-r} r^2 dr \int_0^{2\pi} d\phi \int_{-1}^{+1} (9c^4 - 6c^2 + 1) dc = 1$$

or

$$32\pi N^2/5 = 1$$

$$N = \sqrt{\frac{5}{32\pi}} \qquad \textbf{[4]}$$

Expectation value: We require

$$\int \Psi^* r^2 \Psi d\tau.$$

Integral is the same but with the radial integral replaced by $\int e^{-r} r^4 dr = 4!$. Answer is

$$N^2 \times 16\pi \times 24/5 = 12 \qquad \textbf{[3]}$$

(length units squared)

2. (Bookwork.)

Two observables are Compatible if they can be assigned precise values simultaneously. **[1]**

Consider two different observables represented by Hermitian operators $\hat{A}$ and $\hat{B}$. A precise value λ is assignable to $\hat{A}$ only if the system is in an eigenstate of $\hat{A}$ with eigenvalue λ. Likewise, a precise value μ can be assigned to $\hat{B}$ only if it is in an eigenstate of $\hat{B}$ with eigenvalue μ. Therefore $\hat{A}$ and $\hat{B}$ are compatible only when the system is in an eigenstate of both observables, a simultaneous eigenstate. Let this state have state vector $\psi_{\lambda\mu}$. Then

$$\hat{A}\psi_{\lambda\mu} = \lambda\psi_{\lambda\mu} \quad (i)$$

and

$$\hat{B}\psi_{\lambda\mu} = \mu\psi_{\lambda\mu} \quad (ii)$$

Operate on (i) on the left with $\hat{B}$

$$\hat{B}\hat{A}\psi_{\lambda\mu} = \hat{B}\lambda\psi_{\lambda\mu} = \mu\lambda\psi_{\lambda\mu}$$

Operate on (ii) on the left with $\hat{A}$

$$\hat{A}\hat{B}\psi_{\lambda\mu} = \hat{A}\mu\psi_{\lambda\mu} = \lambda\mu\psi_{\lambda\mu}$$

Subtract the equations

$$(\hat{A}\hat{B} - \hat{B}\hat{A})\psi_{\lambda\mu} = (\lambda\mu - \mu\lambda)\psi_{\lambda\mu} = 0.$$

for a non-trivial solution we must have

$$\hat{A}\hat{B} - \hat{B}\hat{A} = 0. \quad \textbf{[5]}$$

3. (Bookwork.) An operator $\hat{A}$ is Hermitian if for suitable functions f and g

$$\int f^* \hat{A} g d\tau = \int (\hat{A}f)^* g d\tau \quad \textbf{[2]}$$

Consider two eigenstates i and j:

$$\hat{A}\psi_i = \lambda_i \psi_i \quad (1)$$

$$\hat{A}\psi_j = \lambda_j \psi_j \quad (2)$$

Multiply (1) by ψ_j^* on the left and integrate:

$$\int \psi_j^* \hat{A} \psi_i d\tau = \int \psi_j^* \lambda_i \psi_i d\tau = \lambda_i \int \psi_j^* \psi_i d\tau \quad (3)$$

Multiply the complex conjugate of (2) by ψ_i on the left and integrate :

$$\int (\hat{A}\psi_j)^* \psi_i d\tau = \lambda_j^* \int {\psi_j}^* \psi_i d\tau \quad (4)$$

But $\hat{A}$ is Hermitian, so the LHS is

$$\int \psi_j^* \hat{A} \psi_i d\tau$$

Subtract (3) and (4), and since their LHS are identical,

$$(\lambda_i - \lambda_j^*) \int \psi_j^* \psi_i d\tau = 0.$$

Case $i = j$: Since the integral $\int \psi_i^* \psi_i d\tau$ is non-zero.

$$\lambda_j = \lambda_j^*$$

and the eigenvalues are real. **[3]**

Case $i \neq j$: Provided $\lambda_i \neq \lambda_j$,

$$\int \psi_j^* \psi_i d\tau = 0$$

and they are orthogonal. **[2]**

4. (From lectures.)

What is meant in quantum mechanics by the phrase **Collapse of the Wave Function**? [2]

A quantum system resides in general in a superposition of eigenstates until a measurement is made. The measurement causes the system to jump irreversibly with a certain probability into one of its eigenstates. The probability of jumping into a particular eigenstate is given by the squared modulus of its probability amplitude. This is known as collapse or reduction of the wave function.

What is the **Copenhagen Interpretation** of quantum mechanics ? [2]

The Copenhagen Interpretation, associated with Nils Bohr, places the emphasis on measurement. According to this doctrine, quantum mechanics cannot answer the question of what is happening in detail in an experiment. But if an experiment is carried out with a full specification of the entire apparatus used, the surrounding environment and the precise procedure adopted then quantum mechanics can predict the probability of a particular outcome, i.e. the result of the experiment.

Explain, giving an example which illustrates your explanation, what is meant in quantum theory by Complementarity. [3]

(a) Complementarity - Mutually exclusive descriptions (e.g. wave, particle) can be applied to a quantum system but not simultaneously. The wave nature and corpuscular nature of a particle are complementary aspects and never come into conflict in an experimental situation. Example - in the double slit experiment we can either observe an interference pattern between waves or we can determine trajectories of particles. But one excludes the other - the determination of trajectories destroys the interference pattern; the creation of an interference pattern precludes a precise particle trajectory.

[Could also cite as an example the beam-splitter experiment with semi-silvered mirrors, which is essentially the same physics.]

5. For eigenvalues λ

$$(\frac{\hbar}{2}\cos\theta - \lambda)(-\frac{\hbar}{2}\cos\theta - \lambda) - \frac{\hbar^2}{4}\sin^2\theta = 0$$

or

$$\lambda^2 - \frac{\hbar^2}{4}(\cos^2\theta + \sin^2\theta) = 0$$

$$\lambda = \pm\frac{\hbar}{2}. \qquad \textbf{[4]}$$

(Bookwork as far as here.) (Unseen problems, next parts)

Spin polarised in $\hat{n}$ direction:

$$\theta = \pi/3$$

Result is spin-up in $z-$direction with probability

$$\cos^2 \pi/6 = 0.75 \qquad \textbf{[1]}$$

(or spin-down in $z-$direction with probability

$$\sin^2 \pi/6 = 0.25)$$

Second measurement immediately after the first: the measured direction of the spin component is the same as that in the first measurement with probability 1 because the wave function has collapsed into this eigenstate as a result of the first measurement. **[2]**

6. (Unseen Problem)

$$\mathbf{H} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\lambda\mathbf{V} = \begin{pmatrix} \lambda & -\lambda \\ -\lambda & 0 \end{pmatrix}$$

From these definitions,

$$E_1 = 0, \quad E_2 = 1$$

$$W_1 = 0 + \lambda + \lambda^2(\frac{1}{0-1}) = \lambda - \lambda^2$$

In a similar fashion

$$W_2 = 1 + \lambda^2 \qquad \textbf{[6]}$$

7.

$$\mathbf{J} \times \mathbf{J} = \mathbf{i\hbar J}$$

$$[J^2, J_z] = 0.$$

[2]

If J_+ and J_- are defined by

$$J_+ = J_x + iJ_y \quad ; \qquad J_- = J_x - iJ_y,$$

show that

$$[J_z, J_+] = \hbar J_+ \quad ; \qquad [J_z, J_-] = -\hbar J_-,$$

and hence that $J_+ \mid j, m >$ and $J_- \mid j, m >$ are proportional to $\mid j, m + 1 >$and $\mid j, m - 1 >$ respectively.

$$J_z(J_x + iJ_y) - (J_x + iJ_y)J_z = J_zJ_x - J_xJ_z + i(J_zJ_y - J_yJ_z)$$

Using the commutation relations this gives

$$i\hbar J_y + i^2\hbar(-J_x)$$

$$= \hbar(J_x + iJ_y) = \hbar J_+.$$

Similarly,

$$J_z(J_x - iJ_y) - (J_x - iJ_y)J_z = J_zJ_x - J_xJ_z - i(J_zJ_y - J_yJ_z)$$

Using the commutation relations this gives

$$i\hbar J_y - i^2\hbar(-J_x)$$

$$= -\hbar(J_x - iJ_y) = \hbar J_-.$$ [2]

$$(J_zJ_+ - J_+J_z) \mid j \ m >= \hbar J_+ \mid j \ m >$$

$$J_zJ_+ \mid j \ m > -J_+m\hbar \mid j \ m >= \hbar J_+ \mid j \ m >$$

$$J_zJ_+ \mid j \ m >= (m+1)\hbar J_+ \mid j \ m >$$

i.e. $J_+ \mid j \ m >$ is an eigenvector of J_z with eigenvalue $m + 1$. In similar fashion, $J_- \mid j \ m >$ is an eigenvector of J_z with eigenvalue $m - 1$.

These relations tell us that $J_+ \mid j, m >$ and $J_- \mid j, m >$ are proportional to $\mid j, m + 1 >$ and $\mid j, m - 1 >$ respectively. [2]

(Bookwork down to here; what follows is an application candidates have not seen before.)

If J = S

$$S_\pm \mid \tfrac{1}{2}, m >= \hbar\sqrt{\tfrac{3}{4} - m(m \pm 1)} \mid \tfrac{1}{2}, m \pm 1 >$$

Applying this formula gives

$$S_+\alpha = 0; \qquad S_+\beta = \hbar\alpha; \qquad S_-\alpha = \hbar\beta; \qquad S_-\beta = 0. \tag{4}$$

Calculating matrix elements $\alpha^\dagger S_\pm \beta$ etc one obtains matrices

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then using

$$2S_x = S_+ + S_- \qquad 2iS_y = S_+ - S_-$$

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \frac{\hbar}{2i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4}$$

$$H = E_0(S_x^2 + S_y^2 - \hbar S_x)$$

$$H = \frac{E_0\hbar^2}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Let $k = E_0\hbar^2/2$ then

$$H = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}$$

To find eigenvalues,

$$(k - x)(k - x) - k^2 = 0$$

$$x(x - 2k) = 0$$

$x = 0$ or $x = 2k$. For $x = 0$, the normalised eigenvector $\begin{pmatrix} c \\ d \end{pmatrix}$ with $c^2 + d^2 = 1$ is found from

$$\begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = 0$$

This gives

$$\mathbf{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $x = 2k$ we obtain

$$\mathbf{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \textbf{[6]}$$

The general form is thus

$$\psi(t) = \frac{A}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{B}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-iE_0\hbar^2 t/\hbar} \quad \textbf{[3]}$$

where A and B are constants.

At $t = 0$,

$$\psi(0) = \frac{A}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{B}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Hence

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

Probability of finding $x = 0$ is 0.5 Probability of finding $x = E_0\hbar^2$ is also 0.5, as calculation will verify. **[3]**

Thus

$$\psi(t) = \tfrac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-iE_0\hbar t}$$

This equals $\psi(0)$ when $e^{-iE_0\hbar t} = 1$ or

$$t = \frac{2n\pi}{E_0\hbar} \quad \textbf{[4]}$$

8. **(From material presented in lectures)**

Principle of Superposition: If the state vector ψ_1 corresponds to one possible state of a quantum system and the state vector ψ_2 to another, then any linear superposition

$$\psi = C_1\psi_1 + C_2\psi_2$$

where C_1 and C_2 are complex constants, is also a state vector corresponding to a possible state of the system. [2]

Entanglement.

This is a property of correlated many-particle systems. The wave function cannot be expressed as a simple product of one-particle functions. Measurements cannot be made on one particle without affecting the others. (Example is the $S = 0$ spin state of two fermions; the separate spin components of each particle are not defined until after a measurement has been made and the two-particle wave function collapses. For full marks, candidates could use this example to illustrate the concept.) [2]

Non-locality. This is a property of entangled states. Action can be transmitted from one place to affect simultaneously the situation at another arbitrarily distant one. [2]

Classic example is a pair of spin-1/2 particles in a state of total spin S=0. The individual spin components are not defined; all that is known is that they are different; one up, the other down. The particles move apart; measurement of the spin component of one of them immediately fixes the spin component of the other, even though they may be separated by a distance such that no subluminary signal could pass between them. The particles are correlated at all separations.

[Could also quote the example given by Einstein, Podolsky and Rosen of a particle separating into two identical fragments which move apart; or that of a pair of polarised photons.]

As above, candidates could use either of these example to explain the concept.

Qubits. Any two-state system can be used to define bits or binary digits by assigning 0 to one of the states and 1 to the other. A quantum system can reside in a superposition of 0 and 1 with different probability amplitudes

$$\Psi = C_1 \mid 1 > + C_2 \mid 0 >$$

This is known as a qubit. [2]

The Bell states are the four superpositions

$$| \Psi^+ >= \frac{1}{\sqrt{2}}(| 1 >| 0 > + | 0 >| 1 >)$$

$$| \Psi^- >= \frac{1}{\sqrt{2}}(| 1 >| 0 > - | 0 >| 1 >)$$

$$| \phi^+ >= \frac{1}{\sqrt{2}}(| 1 >| 1 > + | 0 >| 0 >)$$

$$| \phi^- >= \frac{1}{\sqrt{2}}(| 1 >| 1 > - | 0 >| 0 >)$$

Photon A is in the state $C_1 \mid 1 > +C_2 \mid 0 >$ **and (B,C) are in the state** $\mid \Psi^- >$.

The three-photon state of A,B and C is then

$$| \Psi >= \frac{1}{\sqrt{2}}[C_1 \mid 1 >| 1 >| 0 > +C_2 \mid 0 >| 1 >| 0 > -C_1 \mid 1 >| 0 >| 1 >$$

$$-C_2 \mid 0 >| 0 >| 1 >] \quad [5]$$

The states of A,B can be expressed as linear combinations of Bell states, e.g.

$$| 1 >| 1 >= \frac{1}{\sqrt{2}}(| \phi^+ + | \phi_-)$$

etc

The three-photon state can then be regrouped in terms of Bell states of photons (A and B) and single-photon states of C:

$$| \Psi >= \frac{1}{2}[| \phi^+ > (C_1 \mid 0 > -C_2 \mid 1 >) + | \phi^- > (C_1 \mid 0 > +C_2 \mid 1 >)$$

$$- | \Psi^+ > (C_1 \mid 1 > -C_2 \mid 0 >) - | \Psi^- > (C_1 \mid 1 > +C_2 \mid 0 >)] \quad [5]$$

The probability of finding any of the Bell states of (A,B) on measurement is thus

$$\tfrac{1}{2}^2 = 0.25. \quad [2]$$

The Bell states can be created in the laboratory by e.g. non-linear crystals. An experimenter can switch from one Bell state

to another by performing an operation on one of the qubits: phase shift (change of sign), bit-flip (e.g. $\mid 0 > \rightarrow \mid 1 >$), combined phase shift and bit flip, and the identity operation (do nothing). All four operations can be performed on polarised photons using wave plates (polarisers), mirrors or non-linear crystals.

Alice wants to teleport a teleportee photon A in the unknown quantum state

$$\mid T >= C_1 \mid 1 > + C_2 \mid 0 >$$

where C_1 and C_2 are probability amplitudes satisfying $\mid C_1 \mid^2 + \mid C_2 \mid^2 = 1$, to Bob.

A pair of ancillary photons B and C is prepared in e.g the Bell state Ψ^-. Photon B is sent to Alice and photon C to Bob.

Alice performs a joint measurement on her ancillary photon and the teleportee photon, and obtains one of the Bell states with probability $\frac{1}{4}$. This measurement collapses Bob's ancillary photon C into a well defined state uniquely related to the state of the teleportee $\mid T >$. Alice then transmits the result of her measurement to Bob over a public channel and he then knows which of the four unitary operations (phase shift, bit flip, combined phase shift/bit flip, identity operation) to perform on his photon C to switch its state to that of the original photon $\mid T >$. Suppose Alice, in her Bell state measurement, finds $\mid \Psi^- >$. C is projected into the state

$C_1 \mid 1 > + C_2 \mid 0 >$, **which is the same as $\mid T >$. Alice then tells Bob the result of her measurement and Bob knows he has to do nothing (perform the identity operation) on his photon; it is already an identical copy of the teleportee. Bob's photon may have been projected into the state instantaneously, but Bob doesn't know he has to do nothing until he receives Alice's message via a subluminary signal. If Alice had found the state $\mid \Psi^+ >$, Bob's photon would have been projected into the state $C_1 \mid 1 > - C_2 \mid 0 >$ and on receiving Alice's message he would have had to have performed a phase change operation to recover the teleportee. In this process of teleportation the original photon is destroyed (by Alice's Bell state measurement), but an identical copy created elsewhere. It is not necessary to know anything about the state of the original photon.** [10]

9. *The Hamiltonian operator H describing a quantum mechanical system in spherical polar co-ordinates has a lowest energy eigenvalue E_0. Show, for any normalisable function $F(\mathbf{r})$ that satisfies the boundary conditions appropriate to a bound state, that the expectation value $E(F)$ of H satisfies*

$$E(F) = \frac{\int F(\mathbf{r})^* H F(\mathbf{r}) d\mathbf{r}}{\int F(\mathbf{r})^* F(\mathbf{r}) d\mathbf{r}} \geq E_0.$$

Use the expansion postulate to expand $F(\mathbf{r})$ in the basis formed by the eigenvectors of H, satisfying

$$H\psi_i = E_i\psi_i; \qquad < \psi_i \mid \psi_j >= \delta_{ij} :$$

$$F = \sum_i a_i \psi_i$$

The expectation value of H in state F is

$$< H >= \int F^* H F d\tau$$

Thus

$$< H >= \int \sum_i a_i^* \psi_i^* H \sum_j a_j \psi_j d\tau$$

$$= \int \sum_{i,j} a_i^* a_j \psi_i^* E_j \psi_j d\tau$$

$$= \sum_{i,j} a_i^* a_j E_i \delta_{ij} = \sum_i \mid a_i \mid^2 E_i$$

Now

$$< F \mid F >= \int F^* F d\tau$$

$$= \int \sum_{i,j} a_i^* a_j \psi_i^* \psi_j d\tau$$

$$= \sum_i \mid a_i \mid^2$$

Let E_0 be the lowest eigenvalue of H, i.e. $E_0 \leq E_i$ for all $i \neq 0$. Then

$$\sum \mid a_i \mid^2 E_i \geq \sum_i \mid a_i \mid^2 E_0 = E_0 \sum_i \mid a_i \mid^2$$

Or

$$\int F^* H F d\tau \geq E_0 \int F^* F d\tau$$

$$E_0 \leq \frac{\int F^* H F d\tau}{\int F^* F d\tau} = E(F) \qquad \textbf{[8]}$$

(Bookwork down to here; what follows is new.)

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

$$\int_{-\infty}^{\infty} F^* F dx = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} = I_0$$

We have

$$< H >= \int F^* H F dx = \frac{-\hbar^2}{2m}(-\alpha + \alpha^2 x^2)e^{-\alpha x^2} + \frac{1}{2}m\omega^2 x^2 e^{-\alpha x^2} dx$$

$$= \frac{\hbar^2}{2m}(\alpha I_0 - \alpha^2 I_2 + \frac{1}{2}m\omega^2 I_2$$

where

$$I_2 = \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}\frac{1}{2\alpha}$$

which gives

$$E = \frac{< H >}{I_0} = \frac{\hbar^2}{4m}\alpha + \frac{m\omega^2}{4\alpha}$$

Differentiate wrt α **:**

$$\frac{dE}{d\alpha} = \frac{\hbar^2}{4m} - \frac{m\omega^2}{4\alpha^2} = 0$$

or

$$\alpha^2 = \frac{m^2\omega^2}{\hbar^2}$$

$$\alpha = \pm\frac{m\omega}{\hbar}$$

We must choose the positive root so that F **tends to zero when** $\mid x \mid$ **is large and**

$$F = e^{\frac{-\omega m x^2}{2\hbar}}. \qquad \textbf{[11]}$$

Excited states: The variational method can be used to find approximate values of the energies of excited states. For example, for the first excited state we choose a trial function F_1 **which is orthogonal to the ground state function** F_0**. If only the variational estimate is available for** F_0 **this introduces a further source of approximation.** F_1 **may be made orthogonal by inspection or**

by using a variational principle with constraints, the method of Lagrange multipliers.

For higher excited states the trial function is orthogonalised to the wave functions for all lower states. [3]

For the oscillator, first excited state; F_1 must be an odd function of x and orthogonal to

$$e^{\frac{-\alpha x^2}{2}}$$

A suitable choice is

$$F_1 = xe^{\frac{-\alpha x^2}{2}}$$ [4]

For the second excited state, we need an even function which has two nodes and is orthogonal to bothF_0 and F_1. A choice would be

$$F_2(\alpha, \beta, x) = (\beta x^2 - 1)e^{\frac{-\alpha x^2}{2}}$$ [4]

10. Commutator:

$$2m\hbar\omega(a_-a_+ - a_+a_-) = (p - im\omega x)(p + im\omega x) - (p + im\omega x)(p - im\omega x)$$

Expanding the brackets on LHS and cancelling gives

$$-2im\omega(xp - px)$$

but $xp - px = i\hbar$, **giving**

$$(a_-a_+ - a_+a_-) = 1. \qquad \textbf{[2]}$$

$$[a_+a_-, a_+] = a_+a_-a_+ - a_+a_+a_-$$

using the fact that $a_-a_+ - a_+a_- = 1$,

$$= a_+a_-a_+ - a_+(a_-a_+ - 1) = a_+$$

Similarly

$$[a_+a_-, a_-] = a_+a_-a_- - a_-a_+a_- = (a_-a_+ - 1)a_- - a_-a_+a_- = -a_-. \qquad \textbf{[3]}$$

Eigenvalue problem for N **is**

$$N \mid n >= \lambda_n \mid n >$$

$$[a_+a_-, a_+] \mid n >= [N, a_+] \mid n >= a_+ \mid n >$$
$$(Na_+ - a_+) \mid n >= a_+N \mid n >= \lambda_n a_+ \mid n > .$$

therefore

$$Na_+ \mid n >= (\lambda_n + 1)a_+ \mid n > .$$

also

$$(Na_- - a_-N) \mid n >= -a_- \mid n >$$
$$Na_- \mid n >= a_-N \mid n > -a_- \mid n >= (\lambda_n - 1)a_- \mid n >$$

thus $a_+ \mid n >$ **is proportional to** $\mid n+1 >$ **and** $a_- \mid n >$ **to** $\mid n-1 >$. **[3]**

Let $\mid 0 >$ **be the state vector of the state with lowest eigenvalue. Then** $a_- \mid 0 >= 0$ **since the state cannot be lowered. Therefore**

$$a_+a_- \mid 0 >= N \mid 0 >= 0$$

or $\lambda_0 = 0$. **Now apply the step-up operator** a_+**:**

$a_+ \mid 0 >$ **is proportional to** $\mid 1 >$ **. so that**

$$Na_+ \mid 0 >= 1 \; a_+ \mid 1 >$$

$$N \mid 1 >= 1 \;\; \mid 1 >$$

i.e next eigenvalue $\lambda_1 = 1$**. Operating on** $\mid 1 >$ **with** a_+ **then gives in similar fashion,** $\lambda_2 = 2$**.. And so on; successive operations with** a_+ **yield the eigenvalues 0,1,2,3,...** **[4]**

(Bookwork down to here; the rest is new to candidates)

Coherent State.

Let

$$\mid \alpha >= \sum_{n=0}^{\infty} C_n \mid n > .$$

Operate on both sides with a_- **and use the equations**

$$a_- \mid \alpha >= \alpha \mid \alpha >$$

and

$$a_- \mid n >= \sqrt{n} \mid n-1 >$$

to obtain

$$\alpha \mid \alpha >= \alpha \sum_{n=0}^{\infty} C_n \mid n >= \sum_{n=1}^{\infty} C_n \sqrt{n} \mid n-1 >$$

Equating coefficients,

$$C_1 = \alpha C_0; \quad \sqrt{2} C_2 = \alpha c_1; \quad \sqrt{3} C_3 = \alpha C_2; \cdots$$

In general

$$\sqrt{n} C_n = \alpha C_{n-1}. \qquad \textbf{[6]}$$

therefore,

$$\sqrt{n(n-1)(n-2)\cdots 3.2.1} C_n = \alpha^n C_0.$$

Hence

$$\mid \alpha >= C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \mid n >$$

Normalise:

$$< \alpha \mid \alpha >= C_0^* C_0 \sum_{m,n} \frac{\alpha^{m+n}}{\sqrt{m!n!}} < m \mid n >= 1.$$

If C_0 **is real,**

$$C_0^2 \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!} = 1$$

or

$$C_0^2 e^{|\alpha|^2} = 1, \qquad C_0 = e^{-|\alpha|^2/2}$$

so that

$$| \alpha >= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \mid n > \qquad \textbf{[6]}$$

probability of finding $\mid k >$ **on measurement is**

$$\mid C_k \mid^2 = e^{-|\alpha|^2} \frac{\alpha^{2k}}{k!} \qquad \textbf{[3]}$$

The average number of quanta is the expectation value of N **in state** α**. Or** $< \alpha \mid N \mid \alpha >$ **. This equals**

$$e^{-|\alpha|^2} \sum_{m,n} \frac{\alpha^{m+n}}{\sqrt{m!n!}} < m \mid N \mid n > .$$

$$= e^{-|\alpha|^2} \sum_{n} \frac{\alpha^{2n}}{n!} n$$

because$N \mid n >= n \mid n >$ **and** $< m \mid n >= \delta_{m,n}$**. This equals**

$$= e^{-|\alpha|^2} \sum_{n=1}^{\infty} \frac{\alpha^{2n}}{(n-1)!}$$

$$= e^{-|\alpha|^2} \alpha^2 e^{+|\alpha|^2}$$

$$= \alpha^2. \qquad \textbf{[3]}$$