UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Physics 2B28: Statistical Thermodynamics and Condensed Matter Physics

COURSE CODE	:	**PHYS2B28**
UNIT VALUE	:	**0.50**
DATE	:	**12-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours 30 Minutes**

04-C1086-3-140

TURN OVER

Answer ALL questions from section A and THREE questions from section B

The numbers in square brackets in the right hand margin indicate the provisional allocation of marks per sub-section of a question.

Boltzmann constant $k = 1.38 \times 10^{-23}$ J K^{-1}
Specific heat of water = 4184 J kg^{-1} K^{-1}
Planck constant $h = 6.627 \times 10^{-34}$ J s
Electronic charge $e = 1.6 \times 10^{-19}$ J
Mass of electron $m_e = 9.1 \times 10^{-31}$ kg
Mass of neutron $m_n = 1.67 \times 10^{-27}$ kg
$\beta = (kT)^{-1}$

SECTION A

1. Explain the following terms used in statistical thermodynamics:

(a) macrostate and accessible microstate [2]
(b) statistical weight [1]
(c) the principle of equal *a priori* probabilities [1]
(d) A system of 4 magnetic dipoles on a square lattice is in a macrostate with 2 dipoles 'up' and 2 dipoles 'down'. Draw all the microstates. [3]

2. (a) Give the Kelvin and the Clausius statements of the Second Law of Thermodynamics. [2]
(b) Show that if the Kelvin statement is untrue, then the Clausius statement is also untrue. [5]

3. (a) One litre of water is heated from 10°C to 90°C by placing it in contact with a large reservoir at 90°C. Calculate the entropy changes of:
(i) the water;
(ii) the reservoir;
(iii) the universe. [3]

(b) One litre of water is heated from 10°C to 90°C by operating a reversible heat engine between it and a reservoir at 90°C. Calculate the entropy changes of:
(i) the water;
(ii) the reservoir;
(iii) the universe. [2]

(c) Explain briefly why the answers to a (iii) and b (iii) differ. [1]

4. Explain what is meant in statistical mechanics by an *ideal classical gas*, and state whether the *internal energy* of an ideal gas depends on its (i) volume, (ii) pressure, and (iii) temperature. [3]

   Deduce the difference in heat capacities at constant pressure and constant volume, for an ideal gas: $C_P - C_V = R$ [4]

5. A particular atom can exist in three magnetic energy states: $E_1 = 1.1 \times 10^{-22}$ J, $E_2 = 1.9 \times 10^{-22}$ J and $E_3 = 3.2 \times 10^{-22}$ J, with degeneracies $g_1 = 1$, $g_2 = 3$ and $g_3 = 5$, respectively. If the atom is in equilibrium at a temperature of 10 K, calculate:

   (a) the value of the partition function $Z(1,V,T)$ [3]
   (b) the probability that the atom has energy E_2 [1]
   (c) the mean energy per atom [3]

6. Two *identical* particles are to be placed in four single-particle states. Two of these states have energy 0, one has energy ε, and the last one has energy 2ε. Calculate the partition function if the particles are (a) fermions, and (b) bosons. [6]

SECTION B

7. (a) State Boltzmann's definition for the *entropy of an isolated system*, explaining the symbols used. [2]

(b) What condition does the entropy of an isolated system satisfy when it is in thermodynamic equilibrium? [1]

(c) An isolated system is partitioned into two sub-systems, 1 and 2. By considering the change in entropy as heat flows from sub-system 1 to sub-system 2, derive the condition that must be satisfied for sub-systems 1 and 2 to be in thermal equilibrium. Show that this leads to the definition of temperature:

$$1/T = (\partial S / \partial E)_{N,V} \qquad [4]$$

(d) A Schottky defect is formed when an atom leaves a perfect crystal lattice and migrates to the surface. Using Stirling's formula (see note below), show that the configurational entropy of n defects on N lattice sites can be written as:

$$S(n) = k\,[\,N \ln N - n \ln n - (N - n) \ln (N - n)] \qquad [4]$$

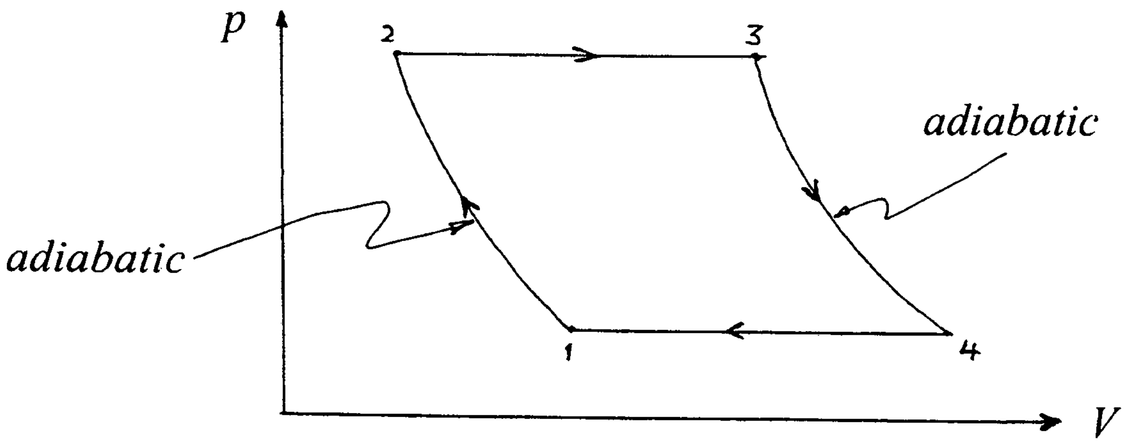
If the energy of formation of a single defect is ε, show that at a temperature T, the equilibrium concentration of defects is approximately:

$$(n / N) = \exp(-\varepsilon / k\,T) \qquad [5]$$

(e) According to one theory, melting occurs when a substance contains 0.1% of vacancy defects. Consider whether this theory can satisfactorily explain the temperatures T_m at which Cu and Pt melt, given that:
for Cu, $\varepsilon = 1.07$ eV and $T_m = 1356$ K;
for Pt, $\varepsilon = 1.3$ eV and $T_m = 2046$ K. [4]

[note Stirling's formula for large N: $\ln(N!) \sim N \ln N - N$]

8. A hypothetical engine, with an ideal monatomic gas as its working substance, operates reversibly in the cycle shown below. At the points 1, 2, 3, 4 in the cycle the pressure, volume and temperature of the gas are (p_1,V_1,T_1), (p_2,V_2,T_2), (p_3,V_3,T_3), (p_4,V_4,T_4), respectively.



(a) Give an expression for the heat absorbed, and the heat rejected, during one cycle, in terms of the temperatures of the gas at the points during the cycle; [2]

(b) Deduce the work done during one cycle; [2]

(c) Show that the efficiency of the engine is

$$\eta = 1 - \left(\frac{p_1}{p_2}\right)^{\frac{\gamma-1}{\gamma}}$$ [10]

(d) If p_1 is 1 atmosphere, what minimum value of p_2 is required for the efficiency to be equal to 40%? [2]

(e) An inventor claims to have developed a cyclic heat engine which exchanges heat between reservoirs at 300K and 540K, producing 450 J of work per 1000J extracted from the hot reservoir. Is this claim feasible? [4]

9. (a) Consider an ideal gas of *photons* in a cavity of volume V and temperature T. The average occupation number of a photon state of energy ε_r is $[\exp(\beta\varepsilon_r) - 1]^{-1}$.
Deduce Planck's law for the distribution $u(\omega, T)d\omega$ of radiation energy in the cavity as a function of angular frequency ω, explaining any assumptions you use:

$$u(\omega, T) = \hbar\, \omega^3 / \pi^2 c^3 \,[\, \exp(\beta\hbar\omega) - 1] \qquad [11]$$

Sketch $u(\omega, T)$ as a function of ω, for T=3000 K and for T=6000 K, marking the visible region of the spectrum on your plot. [2]

What is the significance of the area under each curve? Show that the total energy density of black-body radiation at a temperature T is proportional to T^4. [3]

(b) Stefan's law states that the rate at which radiation is emitted through an area A from a block body at temperature T is given by $A\sigma T^4$, where $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}.

Assuming the Sun is a black body at a temperature T = 5800 K, use Stefan's law to calculate the total radiant energy emitted by the Sun. Show that the rate at which radiant energy reaches the Earth is about 1.4 kW m^{-2}.
[radius of Sun = 7×10^8 m; Earth – Sun distance = 1.5×10^{11} m] [4]

10. For an ideal gas of N identical non-interacting fermions at T = 0 K, explain what is meant the *Fermi energy* ε_F, and the *Fermi temperature* T_F. [2]

State the Fermi-Dirac distribution for the average occupation number of single particle states, $n(\varepsilon)$, as a function of ε, and sketch the distribution at T=0 K. Mark ε_F on your diagram. [2]

Consider an ideal Fermi-Dirac gas of N electrons of mass m_e in a volume V. Show that the Fermi energy can be written as:

$$\epsilon_F = \frac{h^2}{2m_e}\left(\frac{3N}{8\pi V}\right)^{2/3}$$

[8]

Show also that the average energy of the total N electrons is

$$\bar{E} = \frac{3}{5}N\epsilon_F$$

[4]

Calculate the Fermi energy *and* the Fermi temperature for:

(i) conduction electrons in lithium metal, for which $N/V = 4.7 \times 10^{28}$ m^{-3},

(ii) neutrons in a neutron star for the *mass density* $= 1.5 \times 10^{15}$ kg m^{-3} [4]

11.(a) Define the Gibbs function G of a system. Write down an expression for the differential dG. [2]

(b) Deduce the condition for two phases of a homogeneous one-component system to be in equilibrium at constant pressure and temperature. [3]

(c) Derive the Clausius-Clapeyron equation

$$\frac{dp}{dT} = \frac{L}{T\Delta V}$$

relating to equilibrium between phases subject to a first-order phase transition, where L is the latent heat and ΔV is the volume change in passing from one phase to the other. [8]

(d) At T=0.32 K, the melting curve of ^{3}He has zero slope, and the slope becomes negative at lower temperatures. Use the Clausius-Clapeyron equation and the data at the end of the question to calculate the entropy change when one mole of solid ^{3}He melts at T=0.2 K. [2]

(e) Sketch the (p,T) melting curve of ^{3}He, considering carefully its behaviour in the $T = 0$ K limit. Discuss whether you expect the temperature to rise or fall when pressure is applied to convert liquid ^{3}He to solid ^{3}He at $T = 0.2$ K. [5]

[For ^{3}He, the molar volume difference (liquid-solid) = 1.31×10^{-6} m^3 mol^{-1} (independent of temperature), and $dp/dT = -1.3$ MPa K^{-1} at $T = 0.2$ K]