Answer SIX questions from section A and THREE questions from section B.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

SECTION A

[Part marks]

1. Briefly describe the experiment of Davisson and Germer and its implications for our understanding of wave-particle duality. [7]

2. State the Heisenberg Uncertainty Principle. [2]
   Describe the Heisenberg microscope thought experiment and its significance for the Uncertainty Principle. [5]

3. State the **time-independent** Schrödinger equation for the Quantum Harmonic Oscillator (QHO). [2]
   Give 2 ways in which the QHO differs from its classical equivalent. [4]

4. A particle in an infinite well ($V = 0$ if $0 < x < a$ and $V = \infty$ elsewhere) has wavefunction $u(x) = A \sin \frac{2\pi x}{a}$. Work out the quantum energy of this eigenstate. [7]

5. At time $t = 0$, a hydrogen atom has the normalised wavefunction:

$$u(\mathbf{r}) = \sqrt{\frac{4}{3}}\psi_{311}(\mathbf{r}) + \sqrt{\frac{2}{3}}\psi_{321}(\mathbf{r}) + 2\psi_{42-1}(\mathbf{r}),$$

   where the ψ_{nlm} are the normalised eigenfunctions of hydrogen.

   Give $u(\mathbf{r})$ in normalised form, explaining your reasoning. [3]
   If a suitable measurement is carried out, what is the probability that its principal quantum number (n) would be measured to be equal to 3? [2]
   Work out the expectation value of L_z . [2]

6. In the context of dipole radiative transitions of hydrogen, explain briefly the form of the interaction between the atom and the electromagnetic radiation [3]

   State the general **selection rules** on n, l, m, for electric dipole transitions in the hydrogen atom. [3]

SECTION B

7. In Cartesian coordinates the components of the angular momentum vector operator $\hat{\mathbf{L}}$ are related by $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z, \quad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x, \quad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$. In addition,

$$[\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = 0.$$

(a) Explain carefully the physical significance of these results, with reference to the precession model of angular momentum and the significance of spherical harmonics in the quantum theory of angular momentum. [8]

(b) The operators $\hat{L}_z$ and $\hat{L}^2$ can be expressed in terms of the spherical polar angles (θ, ϕ) as

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi},$$

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right].$$

Given the unnormalised spherical harmonic function

$$Y(\theta, \phi) = N\mathrm{e}^{i\phi}\sin\theta,$$

where N is a constant, show that $Y(\theta, \phi)$ is an eigenfunction of $\hat{L}_z$ and $\hat{L}^2$ and determine the corresponding eigenvalues. [8]

(c) Determine the constant N. You may use the result $\int_0^\pi \sin^3 x dx = 4/3$. [4]

8. (a)A wavefunction Ψ is given in terms of a set of orthonormal eigenfunctions, ie $\Psi = \sum_n C_n\phi_n$, for which $C_n = \int \Psi\phi_n^* d\tau$. Discuss the physical significance of this result for the quantum theory of measurement, considering in particular the probability of measuring an arbitrary eigenvalue λ_n as well as the averages obtained after many measurements on identical systems. [8]

(b)An atom of tritium ($Z = 1$) is in a $2p$ state with $m = +1$ when its nucleus suddenly decays into a nucleus of helium ($Z = 2$) without perturbing the extranuclear electron. What will be the probability that we will measure the electron to be in a $2p$ state of helium? [8]

(c) Calculate the expectation value of r imediately after the decay, but before the measurement. [4]

You may assume for an atom of nuclear charge Z, $R_{2p}(r) = \frac{Z^{5/2}}{\sqrt{24}}r\mathrm{e}^{-Zr/2}$ and use the result $\int_0^\infty r^n\mathrm{e}^{-\alpha r}\mathrm{d}r = \frac{n!}{\alpha^{n+1}}$.

9. A particle moves in a one-dimensional potential well, with a potential $V = V_0$ for $|x| > a$ and $V = 0$ elsewhere.

(a)If $E < V_0$, the general solutions for the Time Independent Schrödinger equation take the form $\exp \pm Px$ and $BcosKx$ or $AsinKx$ in the three distinct spatial regions, where K and P are wavenumbers. Discuss the form of these solutions with reference to quantum concepts like tunnelling and parity, identifying the forms appropriate for each region. [8]

(b)The particle is in an odd-parity state, with solution $u(x) = A \sin Kx$ in the well. For this case, show that the wavenumbers obey the relation:

$$\frac{P}{K} = -\cot Ka$$

[6]

(c)By sketching an appropriate graph indicate the values of K corresponding to odd-parity solutions. [3]

(d)Show from your graph that there will be no allowed bound states with odd-parity if the well depth is less than a minimum value V_{min} and give V_{min} in terms of K and a. [3]

10. A particle with energy $E < V_0$ aproaches, from $x = -\infty$, a potential barrier where:

$$\text{Region 1}: \ x \leq 0 \ , \ V = 0$$
$$\text{Region 2}: \ x \geq 0 \ , \ V = V_0$$
$$\text{Region 3}: \ x \geq a \ , \ V = 0$$

(a) Write down model solutions $u(x)$ to the Schrödinger equation in these regions in terms of reflected and transmitted amplitudes R, T; the amplitudes C, D inside the barrier; two wavenumbers k and p relevant to region 1 and 2 respectively. Explain your answers and define the wavenumbers. [7]

(b)Explain the method you would apply to obtain R, T, C, D in terms of k, p and a, giving the 4 relevant equations. [6]

(c) For a high barrier we can approximate:

$$T = \frac{-4ike^{-ika}}{(p - ik)^2 e^{pa}}$$

Evaluate the transmitted current $\frac{\hbar}{2im}(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x})$ [4]

(d) From conservation of current, obtain the reflected current. [3]

11. It can be shown rigorously that, within the dipole approximation, the probability P for a transition between two levels of hydrogen characterised by quantum numbers nlm and $n'l'm'$ obeys the relation:

$$P \propto f^3 \left| \int \Psi_{nlm}(\mathbf{r}) \epsilon.\mathbf{r} \Psi_{n'l'm'}(\mathbf{r}) d^3\mathbf{r} \right|^2$$

where f = frequency and $\Psi_{nlm}(\mathbf{r}) = R_{nl}(r) Y_{lm}(\theta, \phi)$.

(a) Using a simple physical argument, justify this form of the Fermi Golden Rule. [6]

(b) Astronomers observe two spectral lines corresponding respectively to transitions from the $3p$ and the $2p$ states of hydrogen to the ground state.

Calculate the ratio of the corresponding frequencies. [6]

Assuming radiation polarized along the z-axis, $\epsilon.\mathbf{r} = \epsilon r \cos\theta$ and that all the atomic electrons have $m = 0$, calculate the ratio of the corresponding transition probabilities, explaining your reasoning carefully. [8]

You may use

$$\int_0^\infty r^n \mathrm{e}^{-\alpha r} \mathrm{d}r = \frac{n!}{\alpha^{n+1}}.$$

$$R_{1s}(r) = 2\mathrm{e}^{-r}$$

$$R_{2p}(r) = \frac{1}{2\sqrt{6}} r \mathrm{e}^{-r/2}$$

$$R_{3p}(r) = \frac{8}{27\sqrt{6}} (1 - \frac{r}{6}) \mathrm{e}^{-r/3}$$

END OF PAPER