Answer SIX questions from section A and THREE questions from section B.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

SECTION A

[Part marks]

1. Briefly describe the 1991 experiment of Carnal and Mlynek and its implications for our understanding of wave-particle duality. [7]

2. State the **time-dependent** Schrödinger equation for a particle moving in a one-dimensional potential $V(x,t)$. [2]
   If the potential is not explicitly time-dependent, show that the wavefunction, at fixed energy E, takes the general form $u(x)\exp(-iEt/\hbar)$. [5]

3. Describe the tunnelling phenomenon of quantum mechanics. [3]
   Briefly discuss a physical process in which it is important. [4]

4. A particle in an infinite symmetrical well ($V = 0$ if $|x| < a$ and $V = \infty$ elsewhere) has wavefunction $u(x) = A\sin\frac{\pi x}{a}$. Calculate the expectation value of $\hat{P}_x$. [5]
   Sketch the probability of finding the particle at any point in the well. [2]

5. The operators corresponding to components of quantum angular momentum obey the relations $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$. What is the physical significance of this result? [2]
   The ladder operators are defined by $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$. Evaluate $[\hat{L}_+, \hat{L}_-]$. [5]

6. An atom has the normalised angular wavefunction:
   $$u(\theta,\phi) = \frac{1}{\sqrt{3}}Y_{11}(\theta,\phi) + \frac{1}{\sqrt{6}}Y_{21}(\theta,\phi) + \frac{1}{\sqrt{2}}Y_{10}(\theta,\phi),$$
   where the Y are spherical harmonics. Give the probability that its orbital angular momentum squared (L^2) would be measured to be equal to $2\hbar^2$. [3]
   Given the above wavefunction, work out the expectation value of $\hat{L}_z$. [4]

7. A level of a given multi-electron atom is denoted by ${}^3D_1$. Explain the meaning of this symbol in full. [3]
   An excited state of helium has a configuration $2p3d$. Give all the corresponding terms in spectroscopic notation. [4]

8. The Zeeman effect can be modelled by adding a small extra term to the Hamiltonian (in atomic units):

$$\hat{V} = \frac{B}{2}(\hat{L}_z \ + \ 2\hat{S}_z).$$

Explain the physical origin of the terms on the right hand side. [4]

A particular atom has been assigned quantum numbers $S = 0$, $L = 2$. How many Zeeman levels would one expect to observe? Give their quantum numbers. [3]

SECTION B

9. Explain what is meant by the Heisenberg Uncertainty Principle for positions and momenta of a quantum particle. Discuss our current understanding of the Uncertainty Principle by considering in detail *two* of the following: [4]

(a) The Bohr microscope thought experiment

(b) The commutation relations $[\hat{P}_x, \hat{x}]$ and $[\hat{P}_y, \hat{x}]$ explaining their relevance to the Uncertainty Principle.

(c) The probability distribution of position and momentum coordinates of a *gaussian* wavepacket where the probability amplitudes in k is given by $A(k) = \exp\{-\alpha(k - k_0)\}^2$, where $P = \hbar k$ is the momentum and k_0 is a constant. [16]

10. A one-dimensional potential step is defined by:

$$\text{Region 1}: \ x \leq 0 \ , \ V = 0$$
$$\text{Region 2}: \ x > 0 \ , \ V = V_0$$

A particle approaches the step with energy $E < V_0$ from region 1.

(a) Write down model solutions $u(x)$ to the Schrödinger equation in terms of reflected amplitudes R and transmitted amplitudes T as well as two wavenumbers k and q relevant to region 1 and 2 respectively, explaining your answers and defining the wavenumbers. [8]

(b) By matching wavefunctions and derivatives, obtain the transmitted and reflected amplitudes in terms of k and q. [8]

(c) If $|q| = 1/2$ evaluate the ratio $|u(0)|^2 : |u(1)|^2$ explaining the physical significance of this quantity. [4]

11. A particle moves in a one-dimensional potential well, with a potential $V = V_0$ for $|x| > a$ and $V = 0$ elsewhere. If $E < V_0$, the solution for the Time Independent Schrödinger equation (TISE) takes a different form in each of three spatial regions:
Region 1 : $u_1(x) = C \exp(-Px)$
Region 2 : $u_2(x) = D \exp(+Px)$
Region 3 : $u_3(x) = A \cos Kx + B \sin Kx$.
(a) Give the TISE for each of the three regions and identify the range of x corresponding to each of the three regions, justifying the form of the solutions. [10]

(b) Prove that, if $B = 0$, then:

$$\frac{P}{K} = \tan Ka$$

[6]

(c) In this case, what is the maximum number of quantum states with energy $E < (5^2\hbar^2\pi^2)/(8ma^2)$? [4]

12. In the context of dipole radiative transitions of hydrogen, explain briefly the form of the interaction between the atom and the electromagnetic radiation [4]

State the general **selection rules** on the angular momentum quantum numbers lm, for electric dipole transitions in the hydrogen atom and show, on an energy level diagram, the possible transitions among nl states with principal quantum number $1 < n < 5$. [8]

It can be shown rigorously that, within the dipole approximation, the probability P for a transition between two levels of hydrogen characterised by quantum numbers nlm and $n'l'm'$ obeys the relation:

$$P \propto \left| \int Y^*_{lm}(\theta,\phi) V(\theta,\phi) Y_{l'm'}(\theta,\phi) \sin\theta d\phi d\theta \right|^2$$

where V represents the interaction with the radiation and the Y_{lm} are spherical harmonics.

(a) If we choose the photon polarization such that we can take $V(\theta,\phi) = \epsilon \cos\theta$ show that there is a specific selection rule for the m quantum number, $\Delta m = 0$. [4]

(b) Derive the corresponding selection rules for the case $V = \epsilon \cos\theta \sin\phi$. [4]

You can use the fact that the general form of a spherical harmonic is $Y_{lm}(\theta,\phi) = NP_{lm}(\theta) \exp\{im\phi\}$, where N is a normalization constant and the P_{lm} are functions of θ.

13. The radial Schrödinger equation, in atomic units, for an electron in a hydrogen atom for which the orbital angular momentum quantum number, $\ell = 0$, is

$$\left(\frac{d^2}{dr^2} + \frac{2}{r} + 2E\right) F(r) = 0,$$

where E is the total energy.

(a) Put $F(r) = \exp(-r/\nu)y(r)$, where $E = -1/(2\nu^2)$, and show that

$$\frac{d^2 y}{dr^2} = \frac{2}{\nu}\left(\frac{d}{dr} - \frac{\nu}{r}\right) y.$$ [4]

(b) Assuming that $y(r)$ can be expanded as the series

$$y(r) = \sum_{p=0}^{\infty} a_p r^{p+1},$$

where $a_0 \neq 0$, show that the coefficients a_p in the series satisfy the recurrence relation,

$$p(p+1)a_p = \frac{2}{\nu}(p-\nu)a_{p-1}.$$ [8]

(c) Solutions of the radial Schrödinger equation exist which are bounded for all r provided that $\nu = n$, where n is a positive integer. Show that the un-normalized radial function for the $n = 2$ state is

$$F_{2s}(r) = a_0 r \left(1 - \frac{r}{2}\right) e^{-\frac{r}{2}}.$$ [4]

(d) Show that the normalisation constant of the 2s state is $a_0 = \frac{1}{\sqrt{2}}$ and hence determine the expectation value of r^2 for this state. [4]

The following result may be assumed

$$\int_0^\infty r^m e^{-\alpha r}\, dr = \frac{m!}{\alpha^{m+1}}.$$

 END OF PAPER