ASTM003 Angular Momentum and Accretion Processes in Astrophysics

May 2005 Examination Paper Model Answers

A1 (a) Viscosity can produce angular momentum transport. Other mechanisms include: the combined effects of magnetic fields and winds; wave transport.
[Lectures] [3 marks, 1 mark each]

(b) The mean free path is $L = 1/n\sigma = 1/(3 \times 10^{21} \times 10^{-20})$ m $= 0.03$ m
[Unseen] [1 mark]

The kinematic viscosity is $\nu \simeq Lc_s$, where c_s is the sound speed.
For a temperature $T = 5 \times 10^4$ K and gas composed of hydrogen ($\mu = 1$ if ionised),

$$\begin{aligned} c_s &= \sqrt{\frac{8.3 \times 5 \times 10^4}{0.001}}\ \mathrm{m\,s^{-1}} \simeq \sqrt{42 \times 10^7}\ \mathrm{m\,s^{-1}} \\ &\simeq \sqrt{4 \times 10^8}\ \mathrm{m\,s^{-1}} \simeq 2 \times 10^4\ \mathrm{m\,s^{-1}} \end{aligned}$$

(note the conversion to moles involves a factor of 1000). The kinematic viscosity is $\nu \simeq 0.03 \times 2 \times 10^4\ \mathrm{m^2\,s^{-1}} \simeq 600\ \mathrm{m^2\,s^{-1}}$. [Lectures] [3 marks]
The evolutionary timescale is

$$\tau_{ev} \simeq \frac{R_{disc}^2}{3\nu} \simeq \frac{(10^{12})^2}{3 \times 600}\,\mathrm{s} \simeq 1 \times 10^{21}\,\mathrm{s}$$ [3 marks]

Dwarf novae have outburst timescales $\simeq$ days. This is much smaller than the evolutionary timescale if atomic/molecular interactions alone are responsible for the viscosity. [Application of principles from lectures] [2 marks]

(c) The 'alpha' model of viscosity represents the kinematic viscosity as $\nu = \alpha c_s H$ where α is a parameter and H is the half thickness of the accretion disc. This accounts for the viscosity being larger than that contributed by atomic scale processes, particularly the effects of turbulence. [Lectures] [3 marks]

A2 (a) Consider a cylindrical element of gas in the accretion disc of veritcal length $\mathrm{d}z$ and cross-sectional area A. The pressure force on the element in the z-direction is $-A\,\mathrm{d}P$ where $\mathrm{d}P$ is the difference in pressure P across the element.
The gravitational force on the element is $GM\mathrm{d}m/r^2$ where $\mathrm{d}m$ is the mass of the element and r is the distance to the star, with $r^2 = R^2 + z^2$.
The component of the gravitational force in the z direction is

$$-\frac{GM}{r^2}\,\mathrm{d}m\,\frac{z}{r} = -\frac{GM}{r^2}\,(\rho A \mathrm{d}z)\,\frac{z}{r} = -\frac{GM}{r^3}\,\rho\,A\,z\,\mathrm{d}z\ ,$$

where ρ is the density of the gas at the point. For equilibrium,

$$-A\,\mathrm{d}P - \frac{GM}{r^3}\rho\,A\,z\,\mathrm{d}z = 0\ ,$$

which gives the differential equation

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}z} = -\frac{GM}{(R^2+z^2)^{3/2}}\,z\ ,$$

the required result. [Lectures] [7 marks]

For Keplerian rotation, $\Omega^2(R) = \dfrac{GM}{R^3}$. For a thin disc, $R^2 + z^2 \simeq R^2$. So the differential equation becomes,

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}z} = -\frac{GM}{R^3}\,z = -\Omega^2 z\ .$$

For an ideal gas, $P = \mathcal{R}\rho T/\mu$. Since the gas is isothermal, T = constant throughout. Therefore,

$$\frac{\mathrm{d}P}{\mathrm{d}z} = \frac{\mathcal{R}T}{\mu}\frac{\mathrm{d}\rho}{\mathrm{d}z}\ .$$

The differential equation becomes,

$$\frac{1}{\rho}\frac{\mathcal{R}T}{\mu}\frac{\mathrm{d}\rho}{\mathrm{d}z} = -\Omega^2 z\ .$$

Integrating over z at a particular R from a the central plane to a height z,

$$\frac{\mathcal{R}T}{\mu}\int_{\rho_0}^{\rho(z)}\frac{\mathrm{d}\rho'}{\rho'} = -\Omega^2\int_0^z z'\,\mathrm{d}z'\ ,$$

where ρ_0 is the central density in the disc. This gives

$$\frac{\mathcal{R}T}{\mu}\ln\left(\frac{\rho}{\rho_0}\right) = -\frac{\Omega^2 z^2}{2} \qquad \therefore \quad \rho(z) = \rho_0\exp\left(-\frac{\Omega^2\mu z^2}{2\mathcal{R}T}\right)\ .$$

This is of the form $\rho(z) = \rho_0\exp\left(-z^2/2H^2\right)$, where $H = \sqrt{\mathcal{R}T/\mu\Omega^2}$ is a constant at a particular radius R. [Lectures] [7 marks]

(b) The surface mass density is $\Sigma = \int_{-\infty}^{\infty}\rho(z)\,\mathrm{d}z$. Therefore,

$$\begin{aligned}\Sigma &= \int_{-\infty}^{\infty}\rho_0\exp\left(-\frac{z^2}{2H^2}\right)\mathrm{d}z = \rho_0\sqrt{2}\,H\int_{-\infty}^{\infty}\exp\left(-\frac{z^2}{2H^2}\right)\mathrm{d}\left(\frac{z}{\sqrt{2}\,H}\right)\\ &= \rho_0\sqrt{2}\,H\int_{-\infty}^{\infty}\mathrm{e}^{-x^2}\mathrm{d}x = \rho_0\sqrt{2\pi}\,H\end{aligned}$$

using the standard integral. [Lectures] [4 marks]

(c)

$$c_s = \sqrt{\frac{\mathcal{R}T}{\mu}} \quad \text{and} \quad H = \sqrt{\frac{\mathcal{R}T}{\mu\,\Omega^2}}\ . \qquad \therefore \quad H = \frac{c_s}{\Omega}\ .$$

So, $c_s = \Omega\,H$. [Lectures] [2 marks]

A3 (a) A dust particle of mass m_{gr} at a distance z from the central plane of the disc experiences a gravitational force $F = \dfrac{GMm_{gr}}{r^2}$, where M is the mass of the central star, r is the distance from the star, and G is the gravitational constant. If R is the radial distance in the plane of the disc, $r^2 = R^2 + z^2$. The component of the force in the direction perpendicular to the disc is $\dfrac{GMm_{gr}}{r^2}\dfrac{z}{r}$ downwards. The equation of motion is

$$m_{gr}\,\frac{\mathrm{d}v}{\mathrm{d}t} \;=\; \pi a^2 \rho\, c_s v \;-\; \frac{GMm_{gr}}{r^2}\,\frac{z}{r} \;=\; \pi a^2 \rho\, c_s v \;-\; \Omega^2 z\, m_{gr} \;\;,$$

on substituting $\Omega = \sqrt{GM/R^3}$ for Keplerian rotation.
The density of the material of the grain is $\rho_{gr} = m_{gr}/\frac{4}{3}\pi a^3$ for a spherical grain. So

$$\frac{\mathrm{d}v}{\mathrm{d}t} \;=\; \frac{\pi a^2 \rho\, c_s v}{\frac{4}{3}\pi \rho_{gr} a^3} \;-\; \Omega^2 z \;=\; \frac{3\rho\, c_s}{4\rho_{gr} a}\, v \;-\; \Omega^2 z \;\;,$$

the required result. [Lectures] [8 marks]

(b) If the dust grain has reached terminal velocity, $\mathrm{d}v/\mathrm{d}t = 0$. If this happens quickly, $z \simeq H$ still. So the equation in part (a) becomes,

$$0 \;\simeq\; \frac{3\rho c_s}{4\rho_{gr} a}\, v_t \;-\; \Omega^2 H \;. \qquad\qquad \therefore \quad v_t \;\simeq\; \frac{4a\rho_{gr}\Omega^2 H}{3\rho\, c_s}$$

[Lectures, 2 marks]

For a protoplanetary disc we have the standard result $c_s \simeq H\Omega$, and $\rho \simeq \Sigma/2H$.

$$\therefore \quad v_t \;\sim\; \frac{4a\rho_{gr}\,\Omega^2 H}{3\left(\frac{\Sigma}{2H}\right)(H\Omega)} \;=\; \frac{8a\rho_{gr}\,\Omega H}{3\Sigma}$$

[Lectures, 3 marks]

The settling time τ_s will be $\tau_s \;\simeq\; \dfrac{H}{v_t} \;\sim\; \dfrac{3\Sigma}{8a\rho_{gr}\,\Omega H}$.

B1 (a) The constribution to the gravitational potential at a point $(x, y, 0)$ in the rotating frame due to the inverse square law is

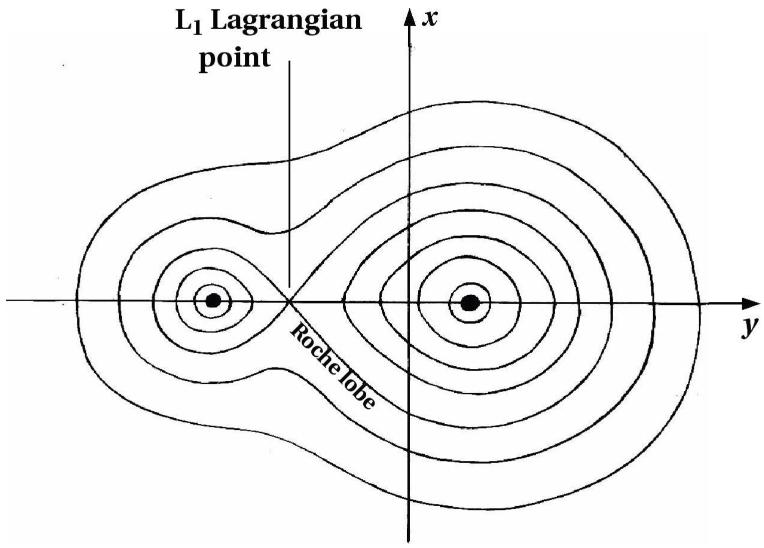
$$\Phi_{grav}(x,y) \;=\; -\;\frac{Gm_2}{\sqrt{(x-x_2)^2+y^2}} \;-\; \frac{Gm_1}{\sqrt{(x-x_1)^2+y^2}}$$

The centrifugal effects can be represented as a centrifugal potential Φ_{cent} in addition to the Φ_{grav} contribution, with $\Phi = \Phi_{grav} + \Phi_{cent}$.
The centrifugal acceleration is $\Omega^2 r$, where r is the distance from the origin (about which the frame is rotating). So $r = \sqrt{x^2+y^2}$. This implies $-\boldsymbol{\nabla}\Phi_{cent} = \Omega^2 r$. This requires $\Phi_{cent}(x,y) \;=\; -\frac{1}{2}\,\Omega^2\, r^2 \;=\; -\frac{1}{2}\,\Omega^2\,(x^2+y^2)$
The total potential in the rotating frame is therefore

$$\Phi(x,y) \;=\; -\;\frac{Gm_2}{\sqrt{(x-x_2)^2+y^2}} \;-\; \frac{Gm_1}{\sqrt{(x-x_1)^2+y^2}} \;-\; \frac{1}{2}\,(x^2+y^2)\,\Omega^2 \;,$$

the required result. [Lectures] [7 marks]

The appearance of the potential in the (x, y) plane is:



[Lectures, with some additional interpretation for unequal masses] [3 marks]

(b) On the x-axis (i.e. $y = z = 0$) we have

$$\Phi(x, y) = -\frac{Gm_2}{|x - x_2|} - \frac{Gm_1}{|x - x_1|} - \frac{1}{2}x^2\,\Omega^2\,,$$

Between the two stars we have $|x - x_2| = x - x_2$ and $|x_1 - x| = x_1 - x$.

$$\therefore \quad \Phi(x) = -\frac{Gm_2}{x - x_2} - \frac{Gm_1}{x_1 - x} - \frac{1}{2}x^2\,\Omega^2\,.$$

Differentiating with respect to x,

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} = \frac{Gm_2}{(x - x_2)^2} - \frac{Gm_1}{(x_1 - x)^2} - x\,\Omega^2\,.$$

At the stationary point (i.e. the L_1 point), $\mathrm{d}\Phi/\mathrm{d}x = 0$. So at this point,

$$\frac{Gm_2}{(x - x_2)^2} - \frac{Gm_1}{(x_1 - x)^2} - x\,\Omega^2 = 0\,.$$

But for orbital motion we have $\Omega^2 = G(m_1 + m_2)/D^3$, where D is the distance between the stars.

$$\therefore \quad \frac{m_2}{(x - x_2)^2} - \frac{m_1}{(x_1 - x)^2} - x\,\frac{m_1 + m_2}{D^3} = 0\,.$$

Put $x = x_2 + r_L$ for this point.

$$\therefore \quad \frac{m_2}{(x_2 + r_L - x_2)^2} - \frac{m_1}{(x_1 - x_2 - r_L)^2} - (x_2 + r_L)\,\frac{m_1 + m_2}{D^3} = 0\,.$$

But $x_1 - x_2 = |x_1| + |x_2| = D$, the separation between the stars, and $x_2 = -m_1 D/(m_1 + m_2)$.

$$\therefore \quad \frac{m_2}{r_L^2} - \frac{m_1}{(D - r_L)^2} - \left(-\frac{m_1}{m_1 + m_2}D + r_L\right)\frac{m_1 + m_2}{D^3} = 0\,,$$

which on rearranging gives,

$$-\frac{m_1}{m_2}D^3 r_L^2 \;+\; D^3(D-r_L)^2 \;-\; \left(\frac{m_1}{m_2}+1\right) r_L^3(D-r_L)^2 \;+\; \frac{m_1}{m_2} D\, r_L^2(D-r_L)^2 \;=\; 0 \;,$$

the required result. [Seen in example problem] [10 marks]

No simple analytic expression can be found for r_L in terms of m_1, m_2 and D. [Unseen] [2 marks]

(c) No account is taken for the Coriolis force in this potential, The Coriolis force depends on the velocity of the gas in the rotating frame, which cannot be represented in this potential. [Unseen] [3 marks]

(d) Roche lobe overflow can occur if:

- one star expands to fill its Roche lobe, due to stellar evolution;
- the orbital separation D decreases due to the loss of angular momentum, for example caused by stellar winds, by tidal effects, or by gravitational radiation.

[Lectures] [3 marks]

(e) In detached binary systems, neither component fills its Roche lobe.
In semi-detached binary systems, one component fills its Roche lobe, the other does not.
In contact binary systems, both components fill their Roche lobes and exist within a common envelope.

[Lectures] [3 marks]

(f) Consider mass transfer from a lobe-filling star of mass m_1 into a ring of radius R_{ring} about a star of mass m_2. Let the distance between the L_1 point and component m_2 be r_L.
The angular velocity of material at the L_1 point about m_2 is Ω. Therefore the specific angular momentum of gas at L_1 relative to m_2 is $j = r_L^2\Omega$. The gas that crosses the L_1 point goes into a circular orbit of radius R_{ring}. Its specifc angular momentum in this orbit is $j = \sqrt{Gm_2R_{ring}}$. From the principle of conservation of angular momentum,

$$\sqrt{Gm_2R_{ring}} \;=\; r_L^2\,\Omega \qquad\qquad \therefore \quad R_{ring} \;=\; \frac{r_L^4\Omega^2}{Gm_2} \;.$$

$$\text{But}\;\; \Omega^2 \;=\; \frac{G(m_1+m_2)}{D} \qquad\qquad \therefore \quad R_{ring} \;=\; \frac{(m_1+m_2)}{m_2}\,\frac{r_L^4}{D^3} \;.$$

[Lectures] [7 marks]

When the masses are equal,

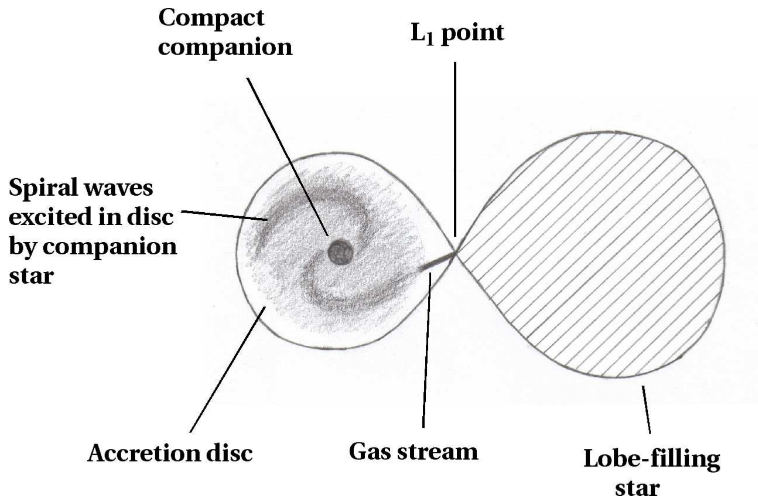
$$R_{ring} \;=\; 2\,\frac{r_L^4}{D^3} \;=\; 2\,\frac{(\frac{1}{2}D)^4}{D^3} \;=\; \frac{D}{8} \;.$$

[Lectures] [3 marks]

(g) If the lobe-filling star is a main sequence star, the other star must be compact (a white dwarf, neutron star or black hole) for an accretion disc to be formed. Otherwise the accretion flow would reach the surface of the star directly. [Lectures] [2 marks]

A cataclysmic variable normally consists of a low mass main sequence star and a white dwarf. [Lectures] [2 marks]

(h)



[Lectures] [5 marks]

B2 (a) The gravitational potential energy of a mass m of material at a radial distance R from a neutron star of mass M is $-GMm/R$. Therefore the energy released when material of mass m is brought from infinity to a radius R is GMm/R.

$$\therefore \quad \frac{\text{energy released}}{mc^2} = \frac{GM}{Rc^2} = \frac{6.7 \times 10^{-11} \times 2 \times 10^{30}}{10^4 \times (3 \times 10^8)^2} \simeq 0.15 \ .$$

So the gravitational potential energy is 10–20% of the $E = mc^2$ rest mass energy. This is larger [by more than an order of magnitude] than the energy available from even the complete fusion of H to Fe. [Lectures] [4 marks]

(b) The Eddington limit is imposed by the radiation pressure on the accreting material. If the accretion rate occurs at the Eddington limit, the rate of energy release creates a sufficient radiation flux that the radiation pressure inhibits further accretion. [Lectures] [4 marks]

(c) Consider material at a radial distance r from the central object. The radiation pressure P_{rad} will produce an outwards force that acts against gravity. When the radiation pressure force is equal to the gravitational force,

$$-\frac{GM}{r^2} - \frac{1}{\rho}\frac{\mathrm{d}P_{rad}}{\mathrm{d}r} = 0 \qquad \therefore \quad \frac{\mathrm{d}P_{rad}}{\mathrm{d}r} = -\frac{GM\rho}{r^2} \ .$$

[Lectures] [4 marks]

The luminosity of the central object is $L = 4\pi r^2 F$, where F is the radiative flux (defined as the energy per unit time per unit surface area).

$$\text{But} \quad F = -\frac{4ac}{3\kappa\rho} T^3 \frac{\mathrm{d}T}{\mathrm{d}r} \qquad \therefore \quad L = -\frac{16\pi ac}{3\kappa\rho} r^2 T^3 \frac{\mathrm{d}T}{\mathrm{d}r} \ .$$

$$\text{But} \quad P_{rad} = \frac{aT^4}{3} \qquad \therefore \quad \frac{\mathrm{d}P_{rad}}{\mathrm{d}r} = \frac{a}{3}\frac{\mathrm{d}}{\mathrm{d}r}\left(T^4\right) = \frac{4a}{3} T^3 \frac{\mathrm{d}T}{\mathrm{d}r}$$

$$\text{So} \quad L = -\frac{4\pi c}{\kappa\rho} r^2 \frac{\mathrm{d}P_{rad}}{\mathrm{d}r} \qquad \text{on substituting for} \quad T^3 \frac{\mathrm{d}T}{\mathrm{d}r} \ .$$

[Lectures] [6 marks]

But the luminosity resulting from a mass accretion rate $\dot{m}$ on to a compact object of mass M and radius R_c is

$$L = \frac{GM\dot{m}}{R_c}$$

[Lectures] [4 marks]

$$\therefore \frac{GM\dot{m}}{R_c} = -\frac{4\pi c}{\kappa\rho} r^2 \frac{\mathrm{d}P_{rad}}{\mathrm{d}r}$$

Substituting for $\frac{\mathrm{d}P_{rad}}{\mathrm{d}r} = -\frac{GM\rho}{r^2}$, $\quad \frac{GM\dot{m}}{R_c} = -\frac{4\pi c}{\kappa\rho} r^2 \left(-\frac{GM\rho}{r^2}\right) = \frac{4\pi c}{\kappa} GM$

$$\therefore \quad \dot{m} = \frac{4\pi c R_c}{\kappa}$$

This is the Eddington limited accretion rate $\dot{m}_{Edd}$. So, $\dot{m}_{Edd} = \frac{4\pi c R_c}{\kappa}$, the required result. [Lectures] [5 marks]

If $\quad R_c = 5R_S = \frac{10GM}{c^2}$, $\quad \dot{m}_{Edd} = \frac{40\pi GM}{c\kappa}$, [Lectures] [3 marks]

(d) $\kappa = 0.04\ \mathrm{m^2\ kg^{-1}}$ and $\mathrm{M} = 10^8\ \mathrm{M_\odot}$.

$$\therefore \quad \dot{m}_{Edd} = \frac{40\pi GM}{c\kappa} \simeq \frac{40 \times 3 \times 6.7 \times 10^{-11} \times 10^8}{3 \times 10^8 \times 0.04}\ M_\odot\ \mathrm{s^{-1}}$$

$$\simeq 7 \times 10^{-8}\ M_\odot\ \mathrm{s^{-1}} \simeq 7 \times 10^{-8} \times 3.2 \times 10^7\ M_\odot\ \mathrm{yr^{-1}} \simeq 2\ M_\odot\ \mathrm{yr^{-1}}$$

[Application of principles discussed in lectures] [4 marks]

(e) $\dot{m} \simeq 2\ M_\odot\ \mathrm{yr^{-1}}$, $M = 10^8\ M_\odot$, $R \sim 10^7$ m.

$$\therefore \quad T_{eff} \simeq 1.2 \times 10^7\ (2)^{\frac{1}{4}}\ (10^8)^{\frac{1}{4}} \left(\frac{10^{12}}{10^7}\right)^{-\frac{3}{4}} \simeq 3 \times 10^5\ \mathrm{K}$$

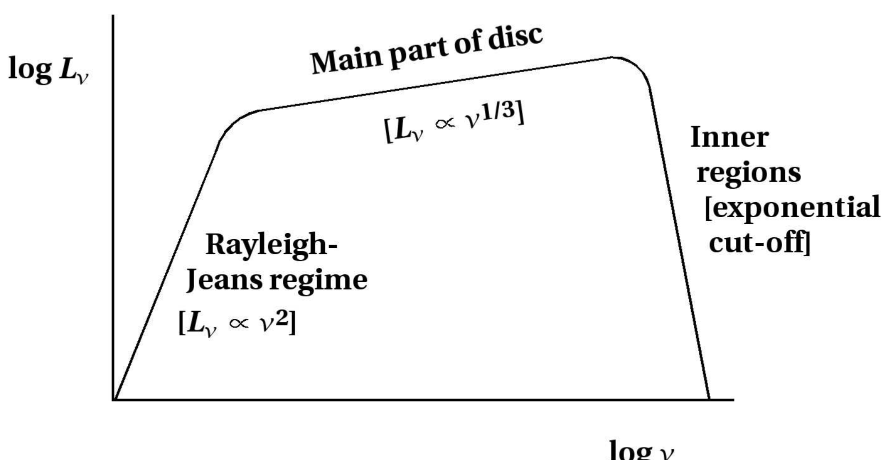
[Principles discussed in lectures] [4 marks]

(f) This corresponds to soft X-rays. The radiation from the inner regions of AGNs includes a strong hard X-ray component, indicative of higher temperatures than predicted here. [Lectures] [2 marks]
This discrepancy can be explained by a hot corona of low density around the main disc. The corona is so hot that it emits the hard X-rays that are observed.
[Lectures] [2 marks]

(g) The temperatures of accretion discs around stellar mass black holes are actually higher than predicted here for the AGNs. They emit hard X-rays.
[Lectures] [2 marks]

(h) The rate of accretion is determined by the rate of gas supply and by the viscosity of the disc. [Viscosity discussed in lectures, gas supply unseen] [2 marks]

(i) Sketch of spectrum:



[Lectures] [4 marks]

B3 (a) Neglecting the pressure contribution,

$$\frac{dv_i}{dt} \;=\; \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j} \quad \text{where} \quad \sigma_{ij} \;=\; \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\boldsymbol{\nabla}.\mathrm{v}\,\delta_{ij}\right) .$$

Consider the dot product $\mathbf{v}\cdot \mathrm{d}\mathbf{v}/\mathrm{d}t$ to calculate the kinetic energy. The dot product is

$$v_i\frac{\mathrm{d}v_i}{\mathrm{d}t} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}v_i^2\right) \;=\; \frac{1}{\rho}\,v_i\,\frac{\partial \sigma_{ij}}{\partial x_j}$$

on substituting for $\mathrm{d}v_i/\mathrm{d}t$. [Lectures] [4 marks]
The kinetic energy per unit volume is $E_V = \frac{1}{2}\,\rho\, v_i\, v_i$. So,

$$\begin{aligned}\frac{\mathrm{d}E_V}{\mathrm{d}t} &= \rho v_i \frac{\mathrm{d}v_i}{\mathrm{d}t} \qquad \text{(for } \rho \text{ independent of time)}\\ &= v_i\,\frac{\partial \sigma_{ij}}{\partial x_j} \;=\; \frac{\partial}{\partial x_j}(v_i\sigma_{ij}) \;-\; \sigma_{ij}\,\frac{\partial v_i}{\partial x_j}\end{aligned}$$

using the product rule. [Lectures] [4 marks]
The $\partial(v_i\sigma_{ij})/\partial x_j$ term expresses the rate of kinetic energy transfer. The $\sigma_{ij}\,\partial v_i/\partial x_j$ term is the rate of energy dissipation due to viscosity per unit volume.
[Lectures] [4 marks]

This energy dissipation term is

$$\begin{aligned}\sigma_{ij}\,\frac{\partial v_i}{\partial x_j} &= \frac{1}{2}\left(\sigma_{ij}\,\frac{\partial v_i}{\partial x_j} \;+\; \sigma_{ij}\,\frac{\partial v_i}{\partial x_j}\right)\\
&= \frac{1}{2}\left(\sigma_{ij}\,\frac{\partial v_i}{\partial x_j} \;+\; \sigma_{ji}\,\frac{\partial v_j}{\partial x_i}\right) \qquad \text{from the symmetry of } \sigma_{ij}\\
&= \frac{1}{2}\left(\sigma_{ij}\,\frac{\partial v_i}{\partial x_j} \;+\; \sigma_{ij}\,\frac{\partial v_j}{\partial x_i}\right) \qquad \text{from} \quad \sigma_{ij} \;=\; \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\boldsymbol{\nabla}.\mathbf{v}\,\delta_{ij}\right)\\
&= \frac{1}{2}\,\sigma_{ij}\left(\frac{\partial v_i}{\partial x_j} \;+\; \frac{\partial v_j}{\partial x_i}\right)\\
&= \frac{1}{2}\,\eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\boldsymbol{\nabla}.\mathbf{v}\,\delta_{ij}\right)\left(\frac{\partial v_i}{\partial x_j} \;+\; \frac{\partial v_j}{\partial x_i}\right) \qquad \text{on subs. for } \sigma_{ij}\\
&= \frac{1}{2}\,\eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j} \;+\; \frac{\partial v_j}{\partial x_i}\right) \qquad \text{using } \boldsymbol{\nabla}.\mathbf{v} = 0\\
&= 2\,\rho\,\eta\left[\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right]\left[\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} \;+\; \frac{\partial v_j}{\partial x_i}\right)\right]\end{aligned}$$

on substituting for $\eta = \rho\nu$ from the definition of the kinematic viscosity ν.
[Lectures] [10 marks]

$$\therefore \quad \sigma_{ij}\,\frac{\partial v_i}{\partial x_j} \;=\; 2\,\rho\,\eta\, e_{ij}\, e_{ij} \qquad \text{where} \quad e_{ij} \;\equiv\; \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} \;+\; \frac{\partial v_j}{\partial x_i}\right) .$$

So the rate of energy dissipation per unit volume is $\epsilon = 2\,\rho\,\eta\,e_{ij}\,e_{ij}$.

[Lectures] [3 marks]

(b) The dissipation per unit area can calculated by integrating perpendicular to the accretion disc.

$$\epsilon_D = \int 2\,\rho\,\nu\,e_{ij}\,e_{ij}\,\mathrm{d}z\,.$$

Substituting for $e_{ij}\,e_{ij} = R^2\,(\mathrm{d}\Omega/\mathrm{d}R)^2$,

$$\epsilon_D = \int 2\,\rho\,\nu\,\frac{R^2}{2}\left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2 \mathrm{d}z = \nu\,R^2\left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2\int\rho\,\mathrm{d}z = R^2\,\nu\,\Sigma\left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2$$

the required result. [Lectures] [5 marks]

(c) In a Keplerian disc, $\dfrac{\mathrm{d}\Omega}{\mathrm{d}R} = -\dfrac{3}{2}\dfrac{\Omega}{R}$. $\quad\therefore\quad \epsilon_D = R^2\,\nu\,\Sigma\left(-\dfrac{3}{2}\dfrac{\Omega}{R}\right)^2 = \dfrac{9}{4}\,\Omega^2\,\nu\,\Sigma\,.$

[Lectures] [4 marks]

(d) Substituting $\nu\,\Sigma = \dfrac{\dot{m}}{3\pi}\left[1-\left(\dfrac{R_*}{R}\right)^{\frac{1}{2}}\right]$ into the expression for ϵ_D,

$$\epsilon_D = \frac{9}{4}\Omega^2\,\frac{\dot{m}}{3\pi}\left[1-\left(\frac{R_*}{R}\right)^{\frac{1}{2}}\right] = \frac{3}{4\pi}\,\dot{m}\left[1-\left(\frac{R_*}{R}\right)^{\frac{1}{2}}\right]\Omega^2$$

$$= \frac{3}{4\pi}\,\dot{m}\left[1-\left(\frac{R_*}{R}\right)^{\frac{1}{2}}\right]\frac{GM}{R^3}\,.$$

[Lectures] [3 marks]

The energy emitted per unit time per unit area of disc is $2\sigma T_{eff}^4$ on assuming all the dissipated energy is radiated as electromagnetic radiation (the factor 2 accounts for the two sides of the disc). So, $\epsilon_D = 2\,\sigma\,T_{eff}^4$ [Lectures] [4 marks]

$$\therefore\quad T_{eff}^4 = \frac{3GM\dot{m}}{8\pi\sigma R^3}\left(1-\left(\frac{R_*}{R}\right)^{\frac{1}{2}}\right)\,.$$

$$\therefore\quad T_{eff} = \left[\frac{3GM\dot{m}}{8\pi\sigma R^3}\left(1-\left(\frac{R_*}{R}\right)^{\frac{1}{2}}\right)\right]^{\frac{1}{4}}\,.$$

[Lectures] [3 marks]

(d) $H = c_S/\Omega = \sqrt{\dfrac{\mathcal{R}T}{\mu}}\sqrt{\dfrac{R^3}{GM}}$ on substituing for c_s and Ω. [Lectures] [1 mark]

$$\therefore\quad H = \sqrt{\frac{\mathcal{R}R^3}{GM\mu}}\sqrt{T} = \sqrt{\frac{\mathcal{R}R^3}{GM\mu}}\left[\frac{3GM\dot{m}}{8\pi\sigma R^3}\left(1-\left(\frac{R_*}{R}\right)^{\frac{1}{2}}\right)\right]^{\frac{1}{8}}$$

on using T_{eff} from part (d) for T. Using the approximation $R_*/R \simeq 0$ for $R \gg R_*$,

$$H \simeq \left[\frac{3\,\mathcal{R}^4\,R^9\,\dot{m}}{8\,\pi\,G^3\,M^3\,\sigma\,\mu^4}\right]^{\frac{1}{8}}\,.$$

[Unseen] [2 marks]

Putting in numerical values,

$$H \simeq \left[\frac{3 \times 8.3^4 \times (1.5 \times 10^{-11})^9 \times 6 \times 10^{14}}{8\,\pi \times (6.7 \times 10^{-11})^3 \times (2 \times 10^{30})^3 \times 5.7 \times 10^{-8} \times (0.001)^4} \right]^{\frac{1}{8}} \mathrm{m}$$

$$\sim \left[2 \times 10^{76} \right]^{\frac{1}{8}} \mathrm{m} \sim 3 \times 10^9 \mathrm{m}$$

[Unseen] [2 marks]

The aspect ratio at a distance $R = 1$ AU $= 1.5 \times 10^{11}$ m is therefore

$$\frac{H}{R} \sim \frac{3 \times 10^9}{1.5 \times 10^{11}} \sim 0.02$$

[Accept any answer in range $0.001 - 0.1$] [Unseen] [1 mark]