Brief summary

Vectors in brief

- Sum or resultant of 2 vectors
- Component representation, e.g. $\underline{a} = \underline{i}a_1 + \underline{j}a_2 + \underline{k}a_3$
- Laws of vector algebra
- Vector multiplication
  - **Scalar or dot product:** $\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$

    Zero for 2 vectors that are perpendicular

    Commute

  - **Vector or cross product:** $\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin\theta \hat{n}$, where $\hat{n}$ is a unit vector such that $\underline{a}$, $\underline{b}$ and $\hat{n}$ form a right handed set.

    Zero for 2 vectors that are parallel

    Anti commute

- Differentiation of vectors

  Product rules of differentiation for vectors

  Be careful about order in vector products!

- Position vector: $\underline{r} = \underline{i}x + \underline{j}y + \underline{k}z$
- Velocity: $\underline{v} = \frac{dr}{dt}$
- Acceleration: $\underline{a} = \frac{dv}{dt}$
- Unit vectors in polar coordinates: $\hat{e_r}$ and $\hat{e_\theta}$
- Express in terms of $\underline{i}, \underline{j}$
- Vel & accel in terms of $\hat{e_r}$ and $\hat{e_\theta}$

  $\underline{v} = \dot{r}\hat{e_r} + r\dot{\theta}\hat{e_\theta}$

  $\underline{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e_r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{e_\theta}$
- Line integrals
- Gradient: $\nabla\phi = \underline{i}\frac{\partial\phi}{\partial x} + \underline{j}\frac{\partial\phi}{\partial y} + \underline{k}\frac{\partial\phi}{\partial z}$
- Curl:

  $\nabla \times \underline{a} = \left(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}\right) \times \left(\underline{i}a_1 + \underline{j}a_2 + \underline{k}a_3\right)$

  $= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$
- An important identity: $\nabla \times \nabla\phi = 0$

Kinematics

- Study of motion without reference to forces.
- Given position vector find velocity and acceleration by differentiation
- Given velocity and acceleration find position vector by integration
- Motion in 1D, 2D or 3D

Newtonian dynamics

- Newton's 3 laws of motion
- Universal law of gravitation $\underline{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$
- Units and dimensions in brief

Applications of Newton's 2nd law

- Motion of particles in 1D

  Under a constant force $F = cons$
  Under resistive force $F = F(v) \propto v^n$
  Under forces of type $F = F(x)$, $F = F(x, v)$
- Motion under the force of gravity:
  Near the surface of the earth where acceleration approximated by constant g

  In general

  Escape velocity

  Black holes
- Motion of particles in 2D: projectiles

Consequences of Newton's Laws

Obtained by integrating 2nd law with respect to t, x

- Definitions of momentum, work, kinetic energy.
- Definition of conservative force: forces for which work done $\int_{\underline{r}_1}^{\underline{r}_2} \underline{F} \cdot d\underline{r}$ is independent of path taken.
- Proof that in 1D forces of the type $F = F(x)$ in one dimension are conservative.
- Proof that the gravitational force is conservative with the corresponding potential $\Phi(r) = -\frac{Gm_1m_2}{r}, \quad r \equiv |\underline{r}|$
- Definition of potential energy.
- The statement of conservation law of energy $\frac{1}{2}m|\underline{v}|^2 + \Phi = KE + PE = E$ a constant
- For motion 1D: $\frac{1}{2}m\dot{x}^2 + \Phi = E$
- The proof that motion near a point of stable equilibrium is SHM.

using the energy equation

Using the energy conservation law to obtain qualitative information about motion.

- Given a potential or force law, use

$$F = -\frac{d\Phi}{dx}, \quad \Phi = -\int F(x)dx$$

- Find equilibrium points: stable and unstable.
- Plot potential $\Phi(x)$.
- Use the energy equation to obtain

$$\dot{x} = \pm\sqrt{\frac{2}{m}\left(E - \Phi(x)\right)}$$

- Motion possible if

  $\Phi(x) \le E$

- Take different values of E and discuss.

SHM

- Proof that motion of a particle of mass m in the neighbourhood of a point of stable equilibrium (taken to be $x = 0$) of any differentiable potential energy function $V(x)$ is periodic, satisfying SHM:

$$m\ddot{x} + kx = 0, \quad \omega^2 = k/m$$

  with period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{V''(0)}}$$

- Be able to derive the general solution of the equation of SHM, $x = a\cos(\omega t - \theta_0)$

Damped SHM

- Understanding the terms in the equation of damped SHM $m\ddot{x} + \alpha\dot{x} + kx = 0$
- Being able to find the general solution of this equation.

- Understanding the nature of solutions in cases of light, large and critical damping.
- Proof that the ratio of neighbouring amplitudes in the case of light damping is

$$\frac{x_{n+1}}{x_n} = -e^{-\frac{\gamma\pi}{\omega}}$$

- Using above ratio to show that the amplitudes of successive oscillations decrease in a geometrical progression.

Forced damped SHM

- General solution sum of particular solution plus particular solution.
- Showing that no matter what the ICs, the oscillations are ultimately governed by the external force with the period of the applied force (ω_1) and not that of the undamped oscillator (ω_0).
- Understanding the phenomenon of resonance

Motion under a central force

- Definition of central force: $\underline{F} = f(r)\hat{\underline{r}}$.
- Definition of angular momentum $\underline{J} = \underline{r} \times m\dot{\underline{r}}$
- Proof that under a central force angular momentum is conserved
- Hence proving that motion under a central force is planar
- Prove that central forces are conservative.
- Hence finding the corresponding potential and the energy conservation law: $\frac{1}{2}m\dot{r}^2 + U(r) = E$, where $U(r) = \frac{J^2}{2mr^2} + \Phi$ is the Effective Potential Energy.
- Being able to find the potential corresponding to a central force.
- Prove that motion under central forces is planar (2D) and conserves angular momentum.
- Obtaining qualitative information about motion using the equation $\frac{1}{2}m\dot{r}^2 + U(r) = E$ above for a given force or potential.

- Knowing the statement (not the proof) of Newton's sphere theorem.
- The derivation of equation of orbit $\frac{d^2u}{d\theta^2} + u = \frac{1}{\ell}$
- Given the equation of orbit being able to find its general solution in the form

$$\frac{1}{r} \equiv u = \frac{1}{\ell}\left[e\cos(\theta - \theta_0) + 1\right]$$

- Understanding what type of orbits corresponding to different values of the eccentricity e.
- Understanding elliptic and circular orbits
- Being able to solve simple examples involving orbits.
- Knowing the statements of the Kepler's laws of planetary motion.