Queen Mary
University of London

M.Sc. EXAMINATION

ASTMO41 Relativistic Astrophysics

Monday, 9 May 2005
10:00-11:30

Solutions

SECTION A

Each question carries 20 marks. You should attempt ALL questions.

1. (a) A spacecraft exploring a planet of normal mass m and radius r moves around the planet along a circular orbit of radius $R = 6r$. Ignoring the transverse Doppler effect, evaluate the redshift z of the radio signal emitted by a probe left on the surface of the planet and received by the spacecraft.

(b) Another spacecraft moves very far from any gravitating bodies with acceleration a. The redshift of a photon emitted at the bottom of the rocket and detected at its top is $z' \approx 10^{-14}$. Evaluate the acceleration a if the height of the rocket is 100 m. [Hint: First calculate the gravitational redshift when $R - r \ll r$ and then apply the equivalence principle.]

A1. Solution

A1(a)

•**[5 Marks]**(*seen similar*)

From conservation of energy, neglecting the transverse Doppler effect, we have

$$h\nu_{ob} - \frac{Gm}{R}\frac{h\nu_{ob}}{c^2} = h\nu_{em} - \frac{Gm}{r}\frac{h\nu_{em}}{c^2}.$$

•**[3 Marks]** (*seen similar*)

Thus

$$\frac{\nu_{ob}}{\nu_{em}} = \frac{1 - \frac{Gm}{rc^2}}{1 - \frac{Gm}{Rc^2}}.$$

•**[5 Marks]** (*unseen*)

Taking into account that in Newtonian limit $Gm/rc^2 \ll 1$, we have

$$\frac{\nu_{ob}}{\nu_{em}} \approx 1 - \frac{Gm}{rc^2}\left(1 - \frac{r}{R}\right) = 1 - \frac{5Gm}{6rc^2},$$

then

$$z = \frac{\nu_{em} - \nu_{ob}}{\nu_{em}} = 1 - \frac{\nu_{ob}}{\nu_{em}} = \frac{GM}{rc^2}(1 - \frac{r}{R}) = \frac{5Gm}{6rc^2}.$$

A1(b)

•**[4 Marks]** (*seen similar*)

If $R = r + h$ and $h \ll r$

$$z = \frac{GM}{rc^2}(1 - \frac{r}{r+h}) \approx \frac{GM}{rc^2}(1 - (1 - \frac{h}{r})) = \frac{GMh}{r^2c^2} = \frac{gh}{c^2},$$

where g is free fall acceleration at the surface of gravitating body.

•**[3 Marks]** (*seen similar*)

According to the equivalence principle

$$z' = \frac{ah}{c^2},$$

then

$$a = \frac{z'c^2}{h} = \frac{10^{-14} \cdot (3 \cdot 10^8)^2 \mathrm{m}^2\mathrm{s}^{-2}}{100\mathrm{m}} = 9\mathrm{m} \cdot \mathrm{s}^{-2}.$$

2. (a) A star forms a black hole of mass M. Show that at the moment when the radius of the star is equal to $10^3 r_g$ its average density is

$$\rho \approx 2 \times 10^{10}\ \mathrm{kg\ m}^{-3} \left(\frac{M}{M_\odot}\right)^{-2}.$$

(b) Using simple Newtonian estimates, show that to an order of magnitude the radius of tidal disruption for a star of mass m and radius r in the gravitational field of a black hole of mass M is

$$R_{TD} \approx r\left(\frac{M}{m}\right)^{1/3}.$$

Compare this radius with the gravitational radius and find the black hole mass for which $R_{TD} = 10^3 r_g$. Give the answer in solar masses.

A2. Solution

A2(a)

•**[8 Marks]** (*seen similar*)

To an order of magnitude

$$\rho = \frac{M}{V} = \frac{M}{\frac{4\pi}{3}10^9 r_g^3} = \frac{3M \cdot 10^{-9}}{4\pi\left(\frac{2GM}{c^2}\right)^3} = \frac{3M_\odot \cdot 10^{-9}}{4\pi\left(\frac{2GM_\odot}{c^2}\right)^3}\left(\frac{M}{M_\odot}\right)^{-2} = \frac{3}{4\pi}\frac{M_\odot \cdot 10^{-9}}{(3\ \mathrm{km})^3}\left(\frac{M}{M_\odot}\right)^{-2} =$$

$$\frac{3 \cdot 2 \times 10^{21}\ \mathrm{kg}}{4 \cdot 3.14 \cdot 3^3 \times 10^{15}\ \mathrm{cm}^3}\left(\frac{M}{M_\odot}\right)^{-2} = \frac{10^9}{10 \cdot 2 \cdot 3}\mathrm{g{\cdot}cm}^{-3}\left(\frac{M}{M_\odot}\right)^{-2} \approx 2{\times}10^7\mathrm{g{\cdot}cm}^{-3}\left(\frac{M}{M_\odot}\right)^{-2}.$$

A2(b)

•**[6 Marks]** (*seen similar*)

To an order of magnitude gravitational force experienced by a particle of mass δm on the surface of the star from the star itself is $F_s \approx Gm\delta m/r^2$, while the tidal force producing a relative acceleration between the the same particle and the centre of the star to an order of magnitude is $F_{TD} \approx GM\delta m r/R^3$, hence defining the tidal radius as the radius at which $F_g \approx F_{TD}$, we have

$$\frac{Gm\delta m}{r^2} \approx \frac{GM\delta m r}{R_{TD}^3},$$

and finally, $R_{TD} \approx r(M/m)^{1/3}$.

•**[6 Marks]** (*seen similar*)
From equality

$$R_{TD} = r_\odot \left(\frac{M}{M_\odot}\right)^{1/3} = 10^3 r_g = 10^3 \frac{2GM}{c^2} = 3\ \text{km} \cdot 10^3 \frac{M}{M_\odot},$$

we have

$$\frac{M}{M_\odot} \approx \left(\frac{r_\odot}{3\ \text{km} \cdot 10^3}\right)^{3/2} = (\frac{7 \cdot 10^5\ \text{km}}{3 \cdot 10^3\ \text{km}})^{2/3} = (2.3 \times 10^2)^{3/3} \approx 3.5 \times 10^3,$$

so $M \approx 3.5 \times 10^3 M_\odot$.

3. (a) An observer moves along a circular orbit of radius r in the equatorial plane ($\theta = \pi/2$) of a rotating black hole. If the gravitational field is described by the Kerr metric, show that this metric can be written in the form

$$ds^2 = \left(g_{00} - \frac{g_{03}^2}{g_{33}}\right) c^2 dt^2 + g_{33} \left(d\phi - \Omega dt\right)^2,$$

where

$$\Omega = -\frac{g_{03}}{g_{33}} = \frac{r_g a}{(r^2 + a^2) r + r_g a^2}.$$

Use the Equivalence Principle to show that the corresponding non-inertial reference frame rotates with angular velocity Ω.

(b) Find the values of r corresponding to the limit of stationarity ($g_{00} = 0$) and the event horizon ($g_{11} = \infty$).

A3. Solution

A3(a)

•**[6 Marks]**(*unseen*)
For $dr = 0$ and $d\theta = \pi/2$

$$ds^2 = g_{00}c^2dt^2 + 2g_{03}cdtd\phi + g_{33}d\phi^2 = g_{00}c^2dt^2 + g_{33}\left(d\phi^2 + \frac{2g_{03}cdtd\phi}{g_{33}} + \frac{g_{03}^2}{g_{33}^2}\right)^2 - \frac{g_{03}^2}{g_{33}}c^2dt^2 =$$

$$= \left(g_{00} - \frac{g_{03}^2}{g_{33}}\right) c^2dt^2 + g_{33}\left(d\phi + \frac{g_{03}}{g_{33}}cdt\right)^2 = \tilde{g}_{00}c^2dt^2 + g_{33}\left(d\phi - \Omega dt\right)^2,$$

where

$$\Omega = -\frac{g_{03}}{g_{33}} = \frac{r_g a}{(r^2 + a^2) r + r_g a^2}.$$

•**[5 Marks]**(*unseen*)
The following transformation of coordinates

$$\tilde{dt} = \sqrt{\left(g_{00} - \frac{g_{03}^2}{g_{33}}\right)} dt,$$

and

$$\tilde{d\phi} = \sqrt{g_{33}} r^{-2}(d\phi - \Omega dt),$$

brings the metric to the form

$$ds^2 = c^2 d\tilde{t}^2 - r^2 d\tilde{\phi}^2,$$

which is locally galilean metric. Hence locally the observer can not discriminate between the Kerr gravitational field and non-inertial frame of reference rotating with angular velocity Ω.

A3(b)

•**[4.5 Marks]**(*seen similar*)

The limit of stationarity: from $g_{00} = 0$ we have $\rho^2 = r_g r$ or

$$r^2 - r r_g + a^2 \cos^2\theta = 0.$$

The larger solution of this equation is

$$r = \frac{r_g}{2} + \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2\cos^2\theta}.$$

•**[4.5 Marks]**(*seen similar*)

The horizon: from $g_{11} = \infty$ we have $\Delta = 0$, thus

$$r^2 - r r_g + a^2 = 0.$$

The larger solution of this equation is

$$r = \frac{r_g}{2} + \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2}.$$

SECTION B

Each question carries 40 marks. Only marks for the best ONE question will be counted.

1. (a) Consider the motion of a particle in the gravitational field of a Schwarzschild black hole. Using the Hamilton-Jacobi equation, show that

$$E\left(1-\frac{r_g}{r}\right)^{-1}\frac{dr}{dt}=c\sqrt{E^2-U_{eff}^2},$$

where U_{eff} is the "effective potential energy":

$$U_{eff}(r)=mc^2\sqrt{\left(1-\frac{r_g}{r}\right)\left(1+\frac{L^2}{m^2c^2r^2}\right)}.$$

Here L is the angular momentum and m is the mass of a particle.

(b) Explain how U_{eff} can be used to find stable and unstable circular orbits.

(c) Show that the radius of the stable circular orbit with angular momentum L is

$$r=\frac{L^2}{m^2c^2r_g}\left[1+\sqrt{1-\frac{3m^2c^2r_g^2}{L^2}}\right].$$

Evaluate the radius of the innermost stable circular orbit.

B1. Solution

B1(a)

•[**4 Marks**](*book work*)

Taking $\theta=\pi/2$ we can write down the Hamilton-Jacobi equation in the Schwarzschild metric as

$$\left(1-\frac{r_g}{r}\right)^{-1}\left(\frac{\partial S}{c\partial t}\right)^2-\left(1-\frac{r_g}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2-\frac{1}{r^2}\left(\frac{\partial S}{\partial\phi}\right)^2-m^2c^2=0.$$

•[**3 Marks**](*book work*)

Then putting $S=-Et+L\phi+S_r(r)$, we have for the radial component of the four-momentum

•[**4 Marks**](*book work*)

$$\frac{\partial S}{\partial r}=p_1=g_{11}p^1=g_{11}\frac{dr}{ds}=\sqrt{\frac{E^2}{c^2}\left(1-\frac{r_g}{r}\right)^{-2}-\left(m^2c^2+\frac{L^2}{r^2}\right)\left(1-\frac{r_g}{r}\right)^{-1}}=$$

$$= \frac{1}{c}\left(1-\frac{r_g}{r}\right)^{-1}\sqrt{E^2-\left[mc^2\left(1+\frac{L^2}{m^2c^2r^2}\right)\left(1-\frac{r_g}{r}\right)\right]^2}.$$

•[**2 Marks**](*book work*)

On other hand

$$\frac{dt}{ds} = p^0 = g^{00}p_0 = g^{00}\left(\frac{\partial S}{\partial t}\right) = -g^{00}E.$$

•[**6 Marks**](*book work*)

Thus

$$\frac{dr}{dt} = \frac{\frac{dr}{ds}}{\frac{dt}{ds}} = \frac{1}{c}\left(1-\frac{r_g}{r}\right)\sqrt{E^2-U_{eff}^2}\frac{1}{E} = \frac{1}{c}\left(1-\frac{r_g}{r}\right)^{-1}\sqrt{E^2-U_{eff}^2},$$

where

$$U_{eff} = mc^2\sqrt{\left(1+\frac{L^2}{m^2c^2r^2}\right)\left(1-\frac{r_g}{r}\right)},$$

hence

$$E\left(1-\frac{r_g}{r}\right)^{-1}\frac{dr}{dt} = c\sqrt{E^2-U_{eff}^2}.$$

B1(b)

•[**3 Marks**](*book work*)

For given radius U_{eff} is equal to the energy of a particle which has the turn point for this r, i.e. $dr/dt = 0$, thus the condition $E > U_{eff}$ determines the admissible range of the motion. The effective potential includes in relativistic manner potential energy plus kinetic energy of non-radial motion, this kinetic energy is determined by angular momentum L.

•[**3 Marks**](*book work*)

All circular orbits are determined by simultaneous solution of the equations

$$U_{eff} = E \quad \text{and} \quad \frac{dU_{eff}}{dr} = 0.$$

B1(c)

•[**2 Marks**](*seen similar*)

From $dU_{eff}/dr = 0$ we have $dU_{eff}^2/du = 0$, where $u = 1/r$.

•[**4 Marks**](*book work*)

Hence

$$-r_g\left(1+\frac{L^2u^2}{m^2c^2}\right) + (1-r_gu)\frac{2L^2u}{m^2c^2} = 0, \quad \text{or} \quad r_gr^2 + 3r_g\left(\frac{L}{mc}\right)^2 - 2\left(\frac{L}{mc}\right)^2 r = 0.$$

•[**3 Marks**](*seen similar*)

[This question continues overleaf . . .]

Solving this equation we have

$$r_{\pm} = \frac{L^2}{m^2c^2r_g} \pm \sqrt{\left(\frac{L^2}{m^2c^2r_g}\right)^2 - \frac{3L^2}{m^2c^2}} = \frac{L^2}{m^2c^2r_g}\left(1 \pm \sqrt{1 - \frac{3r_g^2m^2c^2}{L^2}}\right).$$

•**[1 Mark]**(*seen similar*)
The larger root corresponds to the stable orbit.
•**[2 Marks]**(*seen similar*)
One can see that

$$1 - \frac{3r_g^2m^2c^2}{L^2} > 0.$$

•**[3 Marks]**(*seen similar*)
Hence

$$-\sqrt{3}mcr_g \leq L \leq \sqrt{3}mcr_g.$$

Substituting $L = \sqrt{3}mcr_g$ into equation for the radius of circular orbits, we have for the radius of the innermost stable orbit $r_{lso} = 3r_g$.

2. (a) Following the original Newtonian calculation of Laplace, show that the escape velocity from the surface of a gravitating body is equal to the speed of light if the radius of the body is equal to its gravitational radius. Discuss briefly the difference between a black hole in Newtonian theory and in General Relativity. Explain why the surface $r = r_g$ is called the event horizon. Show that this surface is null.

(b) A supermassive black hole of mass M_{bh} is surrounded by a stellar cluster. Using the result of question 2(b) from Section A, find the black hole mass, M_{crit}, such that for $M_{bh} < M_{crit}$ the tidal disruption takes place outside the black hole horizon. Express the answer in terms of the stellar parameters m_*, r_* and r_{g*}. Estimate M_{crit} if the cluster consists of solar type stars with $m_* = M_\odot$ and $r_* = R_\odot$.

(c) Assume that the luminosity of AGNs and QSOs is generated by be the accretion of gas onto a supermassive black hole, where the gas comes from the tidal disruption of stars. If the luminosity is proportional to the volume V between the event horizon and the sphere of radius R_{TD}, evaluate L as a function of black hole mass and show that the maximum of L is attained at $M_{bh} = \frac{1}{\sqrt{3}} M_{crit}$.

B2. Solution

B2(a)

•[**4 Marks**](*book work*)

The escape velocity is determined from

$$E = \frac{Mv^2}{2} - \frac{GM^2}{R} = 0.$$

Then, if $v = c$,

$$R = \frac{GM^2}{Mv^2/2} = \frac{2GM}{c^2} = r_g.$$

•[**4 Marks**](*book work*)

In the case of "Laplacian black hole" a body with the velocity less than velocity of light, first, moves outward and only after some time starts to move inward, while in the case of black hole in General Relativity motions outward are impossible, because the surface $r = r_g$ is the event horizon and null surface.

•**[4 Marks]**(*book work*)

The surface $r = r_g$ is called the event horizon, because nothing, even light signals, can escape to infinity from this surface. For this reason all events within the horizon can not be observed from outside.

•**[4 Marks]**(*book work*)

For $r = r_g$, $dr = d\theta = d\phi = 0$, hence $ds^2 = g_{00}(cdt)^2 = (1 - r_g/r)c^2dt^2 = 0$. Thus, the world-line of every point on this surface is a null world-line, hence, this surface is null surface.

B2(b)

•**[8 Marks]**(*seen similar*)

From $R_{TD} > r_g$, where r_g is the gravitational radius of the black hole we obtain

$$r_*(\frac{M_{bh}}{m_*})^{1/3} > \frac{2GM_{bh}}{c^2},$$

or

$$M_{bh} < (r_* m_*^{-1/3} c^2/2G)^{3/2},$$

so

$$M_{crit} = c^3(2G)^{-3/2} r_*^{3/2} m_*^{-1/2} = m_*(r_*/r_{g*})^{3/2}.$$

•**[4 Marks]**(*seen similar*)

For solar type stars we have

$$M_{crit} = M_{\odot}(7 \cdot 10^5 km/3km)^{3/2} = 10^8 M_{\odot}.$$

B2(c)

•**[7 Marks]**(*unseen*)

According to the assumption

$$L = kV = \frac{4\pi k}{3}(R_{TD}^3 - r_g^3) = \frac{4\pi k}{3}(r_*^3[\frac{M_{bh}}{m_*}) - (\frac{2GM_{bh}}{c^2})^3]$$

$$= \frac{4\pi k}{3} r_*^3 (\frac{M_{bh}}{m_*})[1 - (\frac{M_{bh}}{M_{crit}})^2]$$

$$\sim M_{bh}[1 - (\frac{M_{bh}}{M_{crit}})^2].$$

•**[5 Marks]**(*book work*)

If $x = M_{bh}/M_{crit}$, then

$$L \sim x(1 - x^2)$$

and

$$L_x^{'} = 1 - 3x^2 = 0$$

gives $x = 1/\sqrt{3}$.

3. (a) A binary system consists of an invisible compact object of mass M_x and a visible star of mass M. The period of the orbit is T, the angle between the normal to the plane of the orbit and the line of sight of the observer is i and the projection of the orbital velocity of the visible star on the line of sight is v. Which variables are measured directly and how? If the visible star is periodically eclipsed by the invisible object, what can you say about the orientation of the binary system?

(b) Using Newtonian theory, show that the mass function

$$f \equiv \frac{(M_x \sin i)^3}{(M_x + M)^2} = \frac{v^3 T}{2\pi G}.$$

(c) Observations of three eclipsing binaries give the following velocities and periods:

Binary number	1	2	3
Velocity in km/s	250	500	1000
Period in min	48	64	128

Assume that the invisible compact object is a black hole if its mass exceeds $3M_\odot$ and that all visible stars in the above binaries have masses between $1M_\odot$ and $10M_\odot$. By evaluating the mass function f in each case, determine which of these binaries contains, may contain or does not contain a black hole.

B3. Solution

B3(a)

•**[2 Marks]**(*book work*)

The observable values are T and v.

•**[2 Marks]**(*book work*)

The period T is measured by clocks, the velocity v is obtained from spectroscopic measurements based on Doppler effect.

•**[3 Marks]**(*unseen*)

In the case we can say, that the line of sight is very close to the orbital plane, which means that $\sin i \approx 1$.

B3(b)

•**[4 Marks]**(*book work*)

We need solve the following system if equations:

$$rM = r_x M_x,$$

$$\omega^2 r_x = GM(r_x + r)^{-2},$$

$$\omega^2 r = GM_x(r_x + r)^{-2},$$

$$v = \omega r \sin i.$$

•**[8 Marks]**(*book work*)

 [This question continues overleaf . . .]

Summing the second with the third, we have

$$\omega^2(r_x + r) = G(M + M_x)(r_x + r)^{-2},$$

and

$$r_x + r = \left[\frac{G(M + M_x)}{\omega^2}\right]^{1/3},$$

$$r_x = r\frac{M}{M_x},$$

$$r(1 + \frac{M}{M_x}) = \left[\frac{G(M + M_x)}{\omega^2}\right]^{1/3},$$

$$v = \omega r \sin i = \omega \sin i \frac{M_x}{M + M_x}\left[\frac{G(M + M_x)}{\omega^2}\right]^{1/3},$$

$$= (G\omega)^{1/3} \sin i \frac{M_x}{(M + M_x)^{2/3}},$$

$$\frac{v^3}{G\omega} = \frac{v^3 T}{2\pi G} = \frac{M_x^3 \sin^3 i}{(M_x + M)^2}.$$

B1(c)

•[**8 Marks**](*unseen*)

For all these objects $\sin i \approx 1$. Introducing $m_x = M_x/M_\odot$,$m = M/M_\odot$, and

$$f = \frac{v^3 T}{2\pi G M_\odot},$$

we have

$$f = \left(\frac{v}{10^3 km/s}\right)^3 \left(\frac{T}{10^3 s}\right).$$

For $m_x = 3$ and $m = 1$

$$\frac{m_x^3}{(m_x + m)^2} = \frac{27}{16} \approx 1.7,$$

For $m_x = 3$ and $m = 5$

$$\frac{m_x^3}{(m_x + m)^2} = \frac{27}{169} \approx 0.16.$$

•[**5 Marks**](*seen similar*)

If $f < 0.16$ there is no black hole **(-)**,

If $0.16 < f < 1.7$ there may be a black hole **(?)**,

If $1.7 < f$ there is a black hole **(+)**.

•[**5 Marks**](*unseen*)

Object 1: $f \approx 0.25^3 \cdot 2.88 \approx 0.045$, **(-)**.

Object 2: $f \approx 0.5^3 \cdot 3.84 \approx 0.48$, **(?)**.

Object 3: $f \approx 1^3 \cdot 7.68 \approx 7.68$, $(+)$.

•**[3 Marks]**(*unseen*)

Thus, the final answer is:

the object N 3 contains a black hole, the object N 2 might contain a black hole, and the objects N 1 does not contain a black hole.

 [End of examination paper.]