Answer ALL questions from SECTION A and TWO from SECTION B.

The numbers in the square brackets in the right-hand margin indicate the provisional allocation of marks per subsection of a question.

SECTION A.

1. Determine a real positive constant N such that the wave function

$$\psi(\mathbf{r}, t) = Ne^{-r/2}(3\cos^2\theta - 1)e^{i\omega t}$$

given in spherical polar coordinates is normalised to unity in three-dimensional space. **[4]**

Calculate the expectation value of r^2 in the state $\psi(\mathbf{r}, t)$. **[3]**

(You may assume that if n is a positive integer $\int_0^\infty e^{-r}r^n dr = n!$.)

2. What is meant by the statement that two observables are **compatible?**

Show that if $\hat{A}$ and $\hat{B}$ are compatible, they commute. **[6]**

3. Define a **Hermitian operator.**

Show that the eigenvalues of such an operator are real and that the eigenfunctions corresponding to eigenvalues that are different are orthogonal. **[7]**

4. What is meant in quantum mechanics by the phrase **collapse of the wave function**? **[2]**

What is the **Copenhagen interpretation** of quantum mechanics ? **[2]**

Explain, illustrating your explanation by an example, what is meant in quantum theory by **complementarity.** **[3]**

5. In the z-direction basis

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

the component $S_n = \mathbf{S} \cdot \hat{\mathbf{n}}$ of the spin $\mathbf{S}$ of a spin $\frac{1}{2}$ particle in a direction $\hat{\mathbf{n}}$ where $\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is given by

$$S_n = \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

Show that the eigenvalues of S_n are $\pm\hbar/2$. **[4]**

Assume that the normalised eigenvector corresponding to eigenvalue $\hbar/2$ is

$$\mid \chi_+ >= \cos(\theta/2)\alpha + \sin(\theta/2)e^{i\phi}\beta.$$

A beam of electrons is spin polarised with component $+\hbar/2$ in a direction defined by the angles $\theta = \pi/3, \phi = \pi/4$.

What would be the probability of finding $+\hbar/2$ in a measurement of the component of spin in the positive z-direction ? **[1]**

What would be the result of a second measurement of this component immediately after the first? Explain your answer. **[2]**

6. The Hamiltonian matrix of a quantum system is given by

$$\mathbf{H} = \begin{pmatrix} \lambda & -\lambda \\ -\lambda & 1 \end{pmatrix}$$

where λ is a real, positive constant with

$$\lambda << 1.$$

Decompose this Hamiltonian into unperturbed and perturbed parts in the form

$$\mathbf{H} = \mathbf{H_0} + \lambda\mathbf{V}$$

and apply the second-order perturbation theory formula

$$W_n = E_n + \lambda V_{n,n} + \lambda^2 \sum_{m \neq n} \frac{\mid V_{m,n} \mid^2}{E_n - E_m}.$$

to determine the eigenvalues of $\mathbf{H}$ to second-order in λ. **[6]**

SECTION B

7. If $\mathbf{J}$ is a quantum mechanical angular momentum operator, write down the commutation relations

(a) among the Cartesian components of $\mathbf{J}$, (b) between J^2 and J_z. **[2]**

If J_+ and J_- are defined by $J_+ = J_x + iJ_y$ and $J_- = J_x - iJ_y$, show that

$$[J_z, J_+] = \hbar J_+ \quad ; \quad [J_z, J_-] = -\hbar J_-,$$ **[2]**

Hence, if

$$J_z \mid j, m >= m\hbar \mid j, m >$$

show that $J_+ \mid j, m >$ and $J_- \mid j, m >$ are proportional to $\mid j, m+1 >$ and $\mid j, m-1 >$ respectively. **[2]**

In the following you may assume that this proportionality takes the form

$$J_\pm \mid j, m >= \hbar\sqrt{j(j+1) - m(m\pm 1)} \mid j, m \pm 1 > .$$

For a spin-$\frac{1}{2}$ system, $\mathbf{J} = \mathbf{S}$ where

$$S^2 \mid s, m >= s(s+1)\hbar^2 \mid s, m >, \quad S_z \mid s, m >= m\hbar \mid s, m >$$

with $s = \frac{1}{2}$ and $m = \pm\frac{1}{2}$. Using the notation $\mid \frac{1}{2}, \frac{1}{2} >= \alpha$, $\mid \frac{1}{2}, -\frac{1}{2} >= \beta$ show that

$$S_+\alpha = 0; \quad S_+\beta = \hbar\alpha; \quad S_-\alpha = \hbar\beta; \quad S_-\beta = 0.$$ **[4]**

Hence show that the matrices representing S_x and S_y in the basis α, β are

$$\mathbf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \mathbf{S}_y = \frac{\hbar}{2i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ **[4]**

The Hamiltonian of a spin-$\frac{1}{2}$ system is

$$H = E_0(S_x^2 + S_y^2 - \hbar S_x)$$

where E_0 is a real, positive constant. Find the matrix of H and determine its eigenvalues and normalised eigenvectors. **[6]**

Write down the general form of the wave function $\psi(t)$ at time t. **[3]**

If $\psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ find the probability of finding on measurement each of the eigenvalues. **[3]**

Find also the times t at which the system returns to its initial state. **[4]**

8. State the **Principle of Superposition** in quantum mechanics. **[2]**

What are meant in quantum mechanics by

Entanglement ? **[2]**

Non-locality ? **[2]**

A qubit ? **[2]**

The four so-called Bell states of a two-photon system are defined to be

$$| \Psi^{\pm} >= \frac{1}{\sqrt{2}}(| 1 >| 0 > \pm | 0 >| 1 >)$$

$$| \phi^{\pm} >= \frac{1}{\sqrt{2}}(| 1 >| 1 > \pm | 0 >| 0 >)$$

Alice wishes to teleport a photon A in an unknown state

$$| T >= C_1 | 1 > +C_2 | 0 >$$

where C_1, C_2 are complex constants satisfying

$$| C_1 |^2 + | C_2 |^2 = 1$$

to a colleague Bob. A pair of ancillary photons (B,C) is prepared in the Bell state $| \Psi^- >$. What is the resulting state vector of the three-photon system A,B and C? **[5]**

Regroup this state vector in terms of Bell states of the subsystem (A,B) and single-photon states of C. What is the probability of finding the pair (A,B) on measurement in any one given Bell state ? **[5]** **[2]**

Using these results, describe the procedure whereby Alice may teleport her photon to a colleague Bob in a different location. **[10]**

9. The Hamiltonian operator H describing a quantum mechanical system has a lowest energy eigenvalue E_0. Show, for any normalisable function $F(\mathbf{r})$ that satisfies the boundary conditions appropriate to a bound state, that the expectation value $E(F)$ of H satisfies

$$E(F) = \frac{\int F(\mathbf{r})^* H F(\mathbf{r}) d\mathbf{r}}{\int F(\mathbf{r})^* F(\mathbf{r}) d\mathbf{r}} \geq E_0.$$

Explain how this expression can be used to find an approximation to the ground state energy which is an upper limit on its value. **[8]**

Use a trial function of the form

$$F(x, \alpha) = e^{-\alpha x^2/2}$$

where α is a variational parameter, to calculate a variational estimate of the ground state energy of a particle of mass m in a one-dimensional Harmonic oscillator potential of the form

$$V(x) = \frac{1}{2} m \omega^2 x^2$$

By minimising the expectation value $E(\alpha)$ of H with respect to variations in α, show that an upper bound on the ground state energy is

$$E = \frac{1}{2} \hbar \omega.$$ **[11]**

Discuss how the variational method could be extended to find approximate energy eigenvalues for excited states. **[4]**

In the case of the potential discussed above, suggest an appropriate form for the trial function for:

(a) the first excited state. **[3]**

(b) the second excited state. **[4]**

You may assume the standard integral for $n = 0, 1, 2 \cdots$ and $\alpha > 0$,

$$\int_{-\infty}^{+\infty} x^{2n} e^{-\alpha x^2} dx = \frac{\sqrt{\pi}(2n)!}{2^{2n} n! \alpha^{n+1/2}}$$

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10. The creation and annihilation operators a_+ and a_- for a one-dimensional harmonic oscillator of mass m and angular frequency ω are defined by

$$a_+ = \frac{1}{\sqrt{2m\hbar\omega}}(p + im\omega x), \qquad a_- = a_+^\dagger,$$

where x and p are position and momentum operators satisfying $[x, p] = i\hbar$. From these definitions show that the commutator

$$[a_-, a_+] = 1.$$ **[2]**

Show further that if N is the operator defined by $N = a_+ a_-$

$$[N, a_+] = a_+ \quad ; [N, a_-] = -a_-.$$ **[3]**

Using the notation $N \mid n >= \lambda_n \mid n >$ show that

$$Na_\pm \mid n >= (\lambda_n \pm 1)a_\pm \mid n >$$ **[3]**

Show that the eigenvalues λ_n of N are $n = 0, 1, 2....$ and give a physical interpretation of N. **[4]**

A state $\mid \alpha >$ that is an eigenstate of the annihilation operator a_-, i.e. $a_- \mid \alpha >= \alpha \mid \alpha >$ with eigenvalue α is called a coherent state. Expand $\mid \alpha >$ in terms of the complete set of eigenstates of the operator N as

$$\mid \alpha >= \sum_{n=0}^{\infty} C_n \mid n >$$

and show that

$$C_1 = \alpha C_0; \quad C_2 = \frac{\alpha}{\sqrt{2}} C_1; \quad C_3 = \frac{\alpha}{\sqrt{3}} C_2; ....C_n = \frac{\alpha}{\sqrt{n}} C_{n-1}$$ **[6]**

Hence show that the normalised state $\mid \alpha >$ is given by

$$\mid \alpha >= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \mid n > .$$ **[6]**

Find the probability that the coherent state contains k quanta and also the average number of quanta in the coherent state. **[6]**

You should assume that

(i) For any two states $\mid \mu >$ and $\mid \nu >$, then $< \mu \mid a_-^\dagger \nu >=< a_- \mu \mid \nu >$.

(ii) $a_- \mid n >= \sqrt{n} \mid n - 1 >$.

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